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OPERATORS WHOSE ASCENT IS 0 OR 1

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ABSTRACT. An operator T on a Hilbert space H is said to be of ascent 0 or 1 if the null spaces of T and T^2 are equal. Let $\mathcal{A}$ denote the collection of all operators on H which have ascent 0 or 1. The object of this note is to study some properties of operators in $\mathcal{A}$. The main results obtained are the following.

1. The direct sum of a collection of operators is in $\mathcal{A}$ if and only if each of these operators is in $\mathcal{A}$.
2. If $T \in \mathcal{A}$ and T has finite descent, then the range of T is closed.
3. If $T \in \mathcal{A}$ and the range of T is not closed, then T is a commutator, that is, T is expressible in the form $AB-BA$ for some operators A and B on H.
4. The set of all operators in $\mathcal{A}$ with descent 0 or 1 is closed in the norm topology of operators.
5. If $T \in \mathcal{A}$, and T has finite descent and further T^k is compact for some k, then T is a finite-dimensional operator.

Let T be an operator (a bounded linear transformation) on a Hilbert space H. Let $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote its range and null space. The ascent $\alpha(T)$ of T is the least nonnegative integer such that $\mathcal{N}(T^k)=\mathcal{N}(T^{k+1})$ for all $k \geqq \alpha(T)$ and the descent $\delta(T)$ of T is the least nonnegative integer such that $\mathcal{R}(T^k)=\mathcal{R}(T^{k+1})$ for all $k \geqq \delta(T)$. Let $\mathcal{A}$ denote the collection of all operators on H whose ascent is 0 or 1. The object of this note is to obtain some properties of operators of this class.

It is clear that an operator T belongs to $\mathcal{A}$ if and only if

$$\overline{\mathcal{R}(T^*)} = \overline{\mathcal{R}(T^{*2})}$$

where T^* denotes the adjoint of T. We can verify that the class $\mathcal{A}$ includes all invertible operators, all isometric operators, all operators of class $(N; k)$ introduced by Istrăţescu (hence all hyponormal operators) [3] and all quasi-Hermitian operators of Dieudonné. Warner [5] has shown that all operators of the form $\alpha I - T$ where T is symmetrizable and α is a scalar such that im $\alpha \neq 0$ are included in $\mathcal{A}$. It is easy to see that $\mathcal{A}$ is closed under the operation of scalar multi-

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plication and is similarity invariant. Also the product of any two operators in class $\mathfrak{A}$ which commute is in $\mathfrak{A}$.

We now prove the following theorems on operators in class $\mathfrak{A}$.

THEOREM 1. *A direct sum* $T_1 \oplus T_2 \oplus T_3 \oplus \cdots$ *of operators on* H *belongs to* $\mathfrak{A}$ *if and only if* $T_n \in \mathfrak{A}$, *for each* n; $n = 1, 2, 3, \cdots$.

PROOF. If $T_1, T_2, \cdots$ are all operators belonging to $\mathfrak{A}$, then their direct sum taken as the infinite operator matrix

$$\begin{bmatrix} T_1 & 0 & 0 & \cdots \\ 0 & T_2 & 0 & \cdots \\ 0 & 0 & T_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

operating on the space $H \oplus H \oplus H \oplus \cdots$, vanishes at any vector $(x_1, x_2, \cdots)$, $x_i \in H$, whenever its square

$$\begin{bmatrix} T_1^2 & 0 & 0 & & \cdots \\ 0 & T_2^2 & 0 & & \cdots \\ 0 & 0 & T_3^2 & 0 & \cdots \\ 0 & 0 & 0 & T_4^2 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

vanishes at that vector. This proves that the direct sum is in $\mathfrak{A}$. Conversely, if the direct sum is of ascent 0 or 1, then $T_k^2 x_k = 0$, $x_k \in H$, implies

$$\begin{bmatrix} T_1 & 0 & 0 & \cdots & & & \\ 0 & T_2 & 0 & \cdots & & & \\ 0 & 0 & T_3 & 0 & \cdots & & \\ 0 & 0 & 0 & T_4 & \cdots & & \\ \cdot & \cdot & \cdot & \ddots & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & T_k & \\ \cdot & \cdot & \cdot & \cdot & & & \ddots \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ x_k \\ 0 \\ \vdots \end{pmatrix} = 0$$

which gives $T_k x_k = 0$. This proves the theorem.

THEOREM 2. *Let $T \in \mathfrak{A}$ and $\delta(T)$ be finite. Then $\mathfrak{R}(T)$ is closed.*

PROOF. By our hypothesis, it follows that $\alpha(T)$ and $\delta(T)$ are equal and that $H = \mathfrak{N}(T) \oplus \mathfrak{R}(T)$; (see [4, Theorem 5-41-G]). Since the null space $\mathfrak{N}(T)$ is always closed, the closedness of $\mathfrak{R}(T)$ is a consequence of Theorem IV.1.12 of Goldberg [2].

THEOREM 3. *If $T \in \mathfrak{A}$ and $\mathfrak{R}(T)$ is not closed, then T is a commutator.*

PROOF. Assume, to the contrary, that T is not a commutator. Then, by the well-known characterization of commutators given by Brown and Pearcy [1], T can be expressed in the form $\alpha + C$, where α is a nonzero scalar and C is compact. But then, by Theorem 5.5 E of [4], $\alpha(T) = \delta(T)$, whence Theorem 2 implies that $\mathfrak{R}(T)$ is closed. This contradiction proves the theorem.

THEOREM 4. *The set of all operators with ascent and descent* 0 *or* 1 *is closed in the norm topology of operators on H.*

PROOF. We first observe that for any operator A, $\mathfrak{R}(A)$ is closed if and only if $\mathfrak{R}(A^*)$ is closed. Thus if $\mathfrak{R}(A)$ is closed, then A (and hence A^2) has ascent and descent 0 or 1 if and only if A^* (and hence A^{*2}) has ascent and descent 0 or 1. If, now, $\{T_n\}$ is a sequence of operators such that $\alpha(T_n) = \delta(T_n) = 0$ or 1, for each n, such that $\|T_n - T\| \to 0$, then by virtue of Theorem 2, $\mathfrak{R}(T_n^*) = \mathfrak{R}(T_n^{*2})$, so that

$$T^2x = 0 \Rightarrow (Tx, y) = \lim_{n\to\infty} (T_nx, y) = \lim_{n\to\infty} (x, T_n^* y) = \lim_{n\to\infty} (x, T_n^{*2} z)$$

$$= \lim_{n\to\infty} (T_n^2x, z) = (0, z) = 0, \qquad \text{for all } y \in H \Rightarrow Tx = 0.$$

Thus $T \in \mathfrak{A}$. Applying the same argument to T^* instead of T we obtain $\mathfrak{R}(T) = \mathfrak{R}(T^2)$ and we are done.

THEOREM 5. *If T has ascent and descent both equal to* 0 *or* 1 *and T^k is compact for some k, then T is a finite-dimensional operator.*

PROOF. It is clear from the hypothesis that $\alpha(T^k) = \delta(T^k) = 0$ or 1 and that $\mathfrak{R}(T^k) = \mathfrak{R}(T)$. It only remains to observe that in virtue of Theorem 2 and compactness of T^k, $\mathfrak{R}(T^k)$ is finite dimensional.

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REFERENCES

1. A. Brown and C. Pearcy, *Structure of commutators of operators*, Ann. of Math. (2) **82** (1965), 112–127. MR **31** #2612.

2. S. Goldberg, *Unbounded linear operators: Theory and applications*, McGraw-Hill, New York, 1966. MR **34** #580.

3. P. B. Ramanujan, *On operators of class* $(N; k)$, Proc. Cambridge Philos. Soc. **68** (1970), 141–142.

4. A. E. Taylor, *Introduction to functional analysis*, Wiley, New York, 1958. MR **20** #5411.

5. K. K. Warner, *A note on a theorem of Weyl*, Proc. Amer. Math. Soc. **23** (1969), 469–471. MR **40** #3343.

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