PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 3, August 1971

MUTUAL ABSOLUTE CONTINUITY OF SETS OF MEASURES

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ABSTRACT. A theorem slightly stronger than the following is proved: If K is a convex set of (signed) measures that are absolutely continuous with respect to some fixed positive sigma-finite measure, then the subset consisting of those measures in K with respect to which all measures in K are absolutely continuous is the complement of a set of first category in any topology finer than the norm topology of measures. This implies, e.g., that any Banach-space-valued measure μ is absolutely continuous with respect to $|\langle\mu(\cdot), x'\rangle|$ for a norm-dense G_δ of elements x' of the dual of the Banach space.

Recently Rybakov [4] proved, by a direct but rather involved construction, that if (S, Σ) is a measurable space and $\mu:\Sigma\to E$ a countably additive measure with values in a Banach space E, then there exists a linear functional $x_0'\in E'$ such that μ is absolutely continuous with respect to the (variation of the) scalar-valued measure $\langle\mu(\cdot), x_0'\rangle$. This note gives a rather less involved proof of a stronger result, as indicated in the abstract above.

To fix our ideas and notation, let (S, Σ, λ) be a σ-finite measure space; since there is a strictly positive function in $\mathcal{L}^1(\lambda)$, there is no loss of generality in taking λ finite, which we shall do. We shall denote by P_n ($n=1, 2, \cdots, \infty$) the set of all n-tuples $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \cdots)$ in $\boldsymbol{R}^n$ (or $l^1_{\boldsymbol{R}}$ if $n=\infty$) for which all $\alpha_i>0$, and by Q_n ($n=1, 2, \cdots, \infty$) the subset of P_n for which $\sum_i \alpha_i=1$. It is obvious that the P_n's and Q_n's are G_δ's in their ambient spaces $\boldsymbol{R}^n$ and $l^1_{\boldsymbol{R}}$, and thus their topologies can be generated by complete metrics; in particular, they are Baire spaces (spaces for which the Baire category theorem holds).

Any measure-theoretic terms or notation not otherwise explained agree with the usage of [1]—except that $|\mu|$ is the variation of μ.

LEMMA 1. *For any set $C\subseteq ca(S, \Sigma)$ of measures absolutely continuous with respect to λ, there exists a countable set $\{\mu_i\}_{i\in I}\subseteq C$ such that if $|\mu_i|(A)=0$ for all i, then $|\mu|(A)=0$ for all $\mu\in C$.*

Received by the editors September 24, 1970.

AMS 1970 *subject classifications*. Primary 28A10, 28A45, 46G10; Secondary 47B99.

Key words and phrases. Measures, absolute continuity, vector-valued measures. Choquet simplex, maximal measures.

[1] Partially supported by NSF-GP-21182.

PROOF. Assuming $C \neq \varnothing$, one can find sets $A \in \Sigma$ for which there exist elements $\mu \in C$ such that $|\mu|(A) > 0$ and every $|\mu|$-null subset of A is λ-null; e.g., if $\mu \neq 0$ and f is its Radon-Nikodým derivative with respect to λ, then $A = \{s \in S: |f(s)| \neq 0\}$ has that property. Zorn's lemma gives a maximal disjoint family $\{A_i\}_{i \in I}$ of such sets, with corresponding measures $\{\mu_i\}_{i \in I}$. Since all but countably many A_i's must be λ-null, I is countable; the existence of an $A \in \Sigma$ with $|\mu_i|(A) = 0$ for all $i \in I$ but $|\mu|(A) > 0$ for some $\mu \in C$ would clearly contradict maximality. Q.E.D.

Let $\mathfrak{M}$ denote the topological vector space of all Σ-measurable scalar-valued functions on S under the topology of convergence in λ-measure. It is easy to verify that for any $A \in \Sigma$, $\delta > 0$ and open set U of scalars, the set $\{f \in \mathfrak{M}: \lambda(\{s \in A: f(s) \notin U\}) < \delta\}$ is an open set in $\mathfrak{M}$, and (consequently) the set $\{f \in \mathfrak{M}: \lambda(\{s \in A: f(s) \notin U\}) = 0\}$ is a G_δ in $\mathfrak{M}$.

LEMMA 2. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in $\mathfrak{M}$, and set $B_i = \{s \in S: f_i(s) \neq 0\}$. Then*

(1) *for any finite n, the set of $\alpha \in P_n$ or Q_n for which*

$$\lambda\left(\left\{s \in \bigcup_{i=1}^{n} B_i: \left|\sum_{i=1}^{n} \alpha_i f_i(s)\right| = 0\right\}\right) = 0$$

is a dense G_δ in P_n or Q_n respectively; moreover,

(2) *the set of $\alpha \in Q_\infty$ for which*

$$\lambda\left(\left\{s \in \bigcup_{i=1}^{n} B_i: \left|\sum_{i=1}^{n} \alpha_i f_i(s)\right| = 0\right\}\right) = 0$$

for all finite n is a dense G_δ in Q_∞.

PROOF. For any finite n, the map $(\alpha_1, \alpha_2, \cdots) \to \sum_{i=1}^{n} \alpha_i f_i$ is continuous from $\boldsymbol{R}^n$ or l_R^1 to $\mathfrak{M}$, and thus all the sets in question are G_δ's. To prove the "density" part of assertion (1) for Q_n it clearly suffices to prove the part concerning P_n, and because $P_n = P_{n-1} \times P_1$ a proof by induction on n follows immediately from verification of the case $n = 2$. To check that case we need only observe that the sets $S_\beta = \{s \in B_1 \cup B_2: f_1(s) + \beta f_2(s) = 0\}$ are disjoint for distinct $\beta \in P_1$, and must therefore all be λ-null except for countably many choices of β. It follows that if (α_1, α_2) does not belong to the nowhere-dense set of solutions of $\alpha_2 = \beta\alpha_1$ for one of those countably many β's, then

$$\lambda(\{s \in B_1 \cup B_2: \alpha_1 f_1(s) + \alpha_2 f_2(s) = 0\}) = 0;$$

so the set of (α_1, α_2)'s for which that relation holds contains the complement of a set of first category and is thus dense.

From assertion (1) it follows immediately that for any fixed finite n the set

$$\left\{\alpha \in Q_\infty : \lambda\left(\left\{s \in \bigcup_{i=1}^{n} B_i : \sum_{i=1}^{n} \alpha_i f_i(s) = 0\right\}\right) = 0\right\}$$

is a dense G_δ in Q_∞. The set of $\alpha \in Q_\infty$ for which that relation holds for *all* finite n is thus a countable intersection of dense G_δ's in Q_∞ and thus again dense by the Baire category theorem, a fact which establishes (2). Q.E.D.

We can now prove the principal theorem, stated in somewhat more general form than given in the abstract above.

THEOREM. *Let K be a bounded*[2] *convex set in a locally convex space F, and suppose K is a Baire space. Let $T: K \to ca(S, \Sigma)$ be an affine map continuous in the norm topology of $ca(S, \Sigma)$, and suppose that the measures in $T[K]$ are all absolutely continuous with respect to λ. Then the set of points $x \in K$ for which all measures in $T[K]$ are absolutely continuous with respect to $|Tx|$ is a dense G_δ in K.*

PROOF. Setting $C = T[K]$ in Lemma 1, we can select a sequence $\{x_i\}_{i=2}^{\infty}$ in K which has the property that if $\mu_i = Tx_i$, then $|\mu_i|(A) = 0$ for all i implies $|Tx|(A) = 0$ for all $x \in K$. Let f_i be a Radon-Nikodým derivative of μ_i with respect to λ, $i = 2, 3, \cdots$, let $B_i = \{s \in S : f_i(s) \neq 0\}$, and let $B = \bigcup_{i=2}^{\infty} B_i$. For each $\epsilon > 0$, the set

$$W_\epsilon = \{\nu \in ca(S, \Sigma) : \text{there exists } \delta > 0 \text{ such that } |\nu|(A) < \delta \text{ implies } \lambda(A \cap B) < \epsilon\}$$

is open in the norm topology of $ca(S, \Sigma)$; indeed, if δ has that property for ν and $\|\nu - \pi\| < \delta/2$, then $\delta/2$ has that property for π. Thus $T^{-1}[W_\epsilon] = \{x \in K : \text{there exists } \delta > 0 \text{ such that } |Tx|(A) < \delta \text{ implies } \lambda(A \cap B) < \epsilon\}$ is open in K, and we claim it is dense. To see this, let a point $x_1 \in K$ and a convex zero-neighborhood N in F be given, and let f_1 be a Radon-Nikodým derivative of $\mu_1 = Tx_1$; set $B_1 = \{s \in S : f_1(s) \neq 0\}$. Let $\eta > 0$ be so small that $0 < t < \eta \Rightarrow (1-t)x_1 + tK \subseteq x_1 + N$; this is possible since K is bounded. Applying Lemma 2 to the sequence $\{f_i\}_{i=1}^{\infty}$, we can find an $\alpha \in Q_\infty$ satisfying condition (2) of that lemma and such that $\sum_{i=2}^{\infty} \alpha_i < \eta$. If n is so large that $(\sum_{i=1}^{n} \alpha_i)^{-1}(\sum_{i=2}^{n} \alpha_i) < \eta$ and $\lambda(B \setminus \bigcup_{i=2}^{n} B_i) < \epsilon/2$, then setting γ_i

[2] Alternatives to this hypothesis are possible. E.g., the assertion of the theorem holds if K is metrizable and $0 \in K$, even if K is unbounded.

$=(\sum_{i=1}^n \alpha_i)^{-1}\alpha_i$ we have $\sum_{i=1}^n \gamma_i x_i \in K \cap (x_1+N)$, and $T(\sum_{i=1}^n \gamma_i x_i) = \sum_{i=1}^n \gamma_i \mu_i$ which has Radon-Nikodým derivative $\sum_{i=1}^n \gamma_i f_i$ with respect to λ. Since

$$\lambda\left(\left\{s \in \bigcup_{i=1}^n B_i : \sum_{i=1}^n \gamma_i f_i(s) = 0\right\}\right) = 0$$

(for the γ_i's are proportional to the α_i's), one sees that λ is absolutely continuous with respect to $|\sum_{i=1}^n \gamma_i \mu_i|$ on $\bigcup_{i=2}^n B_i$, i.e., if $H \subseteq \bigcup_{i=2}^n B_i$ then $\lambda(H)$ can be made arbitrarily small by taking $|\sum_{i=1}^n \gamma_i \mu_i|(H)$ sufficiently small. In particular, there exists $\delta > 0$ with the property that for such an H, $|\sum_{i=1}^n \gamma_i \mu_i|(H) < \delta \Rightarrow \lambda(H) < \epsilon/2$; so if $A \in \Sigma$ is such that $|\sum_{i=1}^n \gamma_i \mu_i|(A) < \delta$, then

$$\lambda\left(A \cap \bigcup_{i=2}^n B_i\right) < \epsilon/2 \quad \text{and} \quad \lambda\left(A \cap \left(B \setminus \bigcup_{i=2}^n B_i\right)\right) < \epsilon/2$$

and thus $\lambda(A \cap B) < \epsilon$.

We now need only use the fact that K is a Baire space to conclude that $K_0 = T^{-1}[\bigcap_{k=1}^\infty W_{1/k}]$ is dense in K. It is immediate that K_0 is the set of all $x \in K$ such that $|Tx|(A) = 0$ implies $\lambda(A \cap B) = 0$; but since $S \backslash B$ is null with respect to all $\{\mu_i\}_{i=2}^\infty$ and thus with respect to $|Ty|$ for all $y \in K$, K_0 is the set of all $x \in K$ such that $|Tx|(A) = 0 \Rightarrow |Ty|(A) = 0$ for all $y \in K$. Q.E.D.

A strengthened form of Rybakov's theorem follows at once. Indeed, if $\mu : \Sigma \to E$ is a Banach-space valued vector measure, there is an adjoint linear mapping $T : E' \to ca(S, \Sigma)$ given by $(Tx')(A) = \langle \mu(A), x' \rangle$ and T is continuous in the norm topologies. It is known that there exists a finite positive measure $\lambda \in ca(S, \Sigma)$ with respect to which all the measures Tx' are absolutely continuous (see [**1**, IV.10.5, p. 321] or, for a direct, elegant proof see [**2**, Theorem 3.10, p. 199 ff.]). Taking the set K of our theorem to be the closed unit ball in E' (with the norm topology) and T to be the mapping just described, we see that the set K_0 of x_0' in that unit ball having the property that μ is absolutely continuous with respect to $\langle \mu(\cdot), x_0' \rangle$ is a norm-dense G_δ. This has the amusing consequence that the norm of elements $x \in E$ can be computed by taking the supremum of $|\langle x, x_0' \rangle|$ over $x_0' \in K_0$. For another application of the theorem, suppose X is a simplex in the sense of Choquet [**3**], and let P be the unique linear extension of the mapping which assigns to each positive measure μ on X the maximal measure which represents it. P is norm-decreasing and therefore norm-continuous. Thus if K is a norm-closed convex set of measures on X such that $P[K]$ is absolutely continuous with respect to some fixed positive measure λ, then there is a norm-dense G_δ set $K_0 \subseteq K$ such that every measure in $P[K]$ is absolutely continuous with respect to $|P\mu|$ for each $\mu \in K_0$.

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