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PROPERTY P AND DIRECT INTEGRAL DECOMPOSITION OF W-* ALGEBRAS

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ABSTRACT. If $\mathfrak{A}$ is a W-* algebra on separable Hilbert space H, and if $\mathfrak{A}(\lambda)$ are the factors in the direct integral decomposition of $\mathfrak{A}$, then $\mathcal{P}=\{\lambda \mid \mathfrak{A}(\lambda)$ has property P$\}$ is μ-measurable, and $\mathfrak{A}$ has property P iff $\mathfrak{A}(\lambda)$ has property P μ-a.e.

Let $\mathfrak{A}$ be a W-* algebra on separable Hilbert space H. Let $\mathfrak{A}$ have direct integral decomposition into factors

$$\mathfrak{A} = \int_\Lambda \bigoplus \mathfrak{A}(\lambda)\mu(d\lambda),$$

where K denotes the underlying separable Hilbert space of H. We assume that the reader is familiar with the notation and methods of [**7**, Chapter I] and of [**8**].

We establish the following notation for this paper. $\mathfrak{A}_1$ denotes the unit sphere of $\mathfrak{A}$, and $\mathfrak{U}(\mathfrak{A})$ denotes the unitary operators in $\mathfrak{A}$. $\mathfrak{Z}$ will denote the center of $\mathfrak{A}$ ($\mathfrak{Z}$ consists of all diagonal operators in $B(H)$ [**7**, Theorem I.5.9]). $\mathcal{E}(\mathfrak{A})$ denotes the set of all nonnegative real-valued functions on $\mathfrak{U}(\mathfrak{A})$ which vanish except at a finite number of points and which satisfy $\sum_{U\in\mathfrak{U}(\mathfrak{A})} f(U)=1$. We call the finite set of U such that $f(U)\neq 0$ the *support* of f. We write $f\cdot T=\sum_{U\in\mathfrak{U}(\mathfrak{A})} f(U)UTU^*$ for $T\in B(H)$ (or $T\in B(K)$ if we are dealing with $\mathfrak{A}(\lambda)$).

If L is a separable Hilbert space, let S denote the unit sphere of L and let $\{x_i\}$ be a fixed dense sequence in S. If d denotes the metric of [**7**, Lemma I.4.8] which defines the weak topology on bounded sets in $B(L)$, then, defining $W(A)=d(A,0)$, we have, for a bounded sequence T_n, that $T_n\to 0$ weakly iff $W(T_n)\to 0$. Moreover, if U is unitary, $W_U(A)=W(UAU^*)$ also determines weak convergence to 0. We use this fact below in Lemma 3.

It follows from [**8**, Lemma 1.5] and from the proof of [**8**, Lemma 3.5] that there are countable sequences $A_n\in\mathfrak{A}_1$ ($A_n'\in\mathfrak{A}_1'$, $U_n\in\mathfrak{U}(\mathfrak{A})$) such that for μ-a.e. λ the sequence $A_n(\lambda)$ ($A_n'(\lambda)$, $U_n(\lambda)$) is strong-* dense in $\mathfrak{A}(\lambda)_1$ ($\mathfrak{A}'(\lambda)_1$, $\mathfrak{U}(\mathfrak{A}(\lambda))$). Moreover (see remark after [**8**,

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Lemma 2.2]) we may assume that all operators we deal with are strong-* continuous.

Our aim in this paper is to study property P of $\mathfrak{A}$ in relation to the factors $\mathfrak{A}(\lambda)$.

DEFINITION 1 [7, p. 168]. $\mathfrak{A}$ has *property* P (*property* CP) if for every $T\in B(H)$ (every $T\in \mathcal{Z}'$) the intersection of the weakly closed convex hull of $K(T)=\{UTU^*\mid U\in\mathfrak{U}(\mathfrak{A})\}$ with $\mathfrak{A}'$ is not empty.

For a factor, property CP is simply property P. We shall show that this is always the case, and we use this to prove that $\mathfrak{A}$ has property P iff $\mathfrak{A}(\lambda)$ has property P μ-a.e.

LEMMA 2. *Let* $\mathfrak{B}$ *be an Abelian W-* algebra on separable Hilbert space L. Then* $\mathfrak{B}$ *has property* P.

PROOF. By [7, Lemma 2.1] $\mathfrak{B}$ is generated by a Hermitian operator A. We may assume that $\sigma(A)\subset[0, 1]$. Letting E_t denote the spectral projections of A, it is clear that $\mathfrak{B}$ is generated by the increasing sequence of finite-dimensional *-subalgebras $\mathfrak{B}_m$ generated by $\{E_t\mid t=n2^{-m},\ n=1, 2, \cdots, 2^m\}$. Hence $\mathfrak{B}$ has property P [7, p. 168]. Q.E.D.

LEMMA 3. *If* $\mathfrak{A}$ *has property* CP *then* $\mathfrak{A}$ *has property* P.

PROOF. Suppose $\mathfrak{A}$ has property CP. Let $T\in B(H)$ be given. Since we are dealing with weak convergence on bounded sets of operators, it follows from Lemma 2 (with $\mathfrak{B}=\mathcal{Z}$ and $H=L$) and our hypothesis that there are sequences $f_k\in\mathcal{E}(\mathcal{Z})$ and $g_k\in\mathcal{E}(\mathfrak{A})$ and operators $A\in\mathcal{Z}'$ and $A'\in\mathfrak{A}'$ such that $f_k\cdot T\rightarrow A$ weakly and $g_k\cdot A\rightarrow A'$ weakly. Define W as above for $L=H$, and let $B_k=f_k\cdot T$ and $C_k=g_k\cdot A$. Then $W(B_k-A)\rightarrow 0$ and $W(C_k-A')\rightarrow 0$. It suffices to show that given $\epsilon>0$ there is $h\in\mathcal{E}(\mathfrak{A})$ such that $W(h\cdot T-A')<\epsilon$ in order to conclude that $\mathfrak{A}$ has property P.

Given $\epsilon>0$, we can find g_k such that $W(C_k-A')<\epsilon/2$. Let the support of g_k consist of m operators. Then we can choose f_k so that, for each U in the support of g_k, $W_U(B_k-A)<\epsilon/(2m)$. It follows that

$$W((g_k\cdot B_k)\ -\ A')\ <\ \epsilon.$$

Moreover, if we put $h=g_k * f_k$ (convolution) then $g_k\cdot B_k=h\cdot T$ and $h\in\mathcal{E}(\mathfrak{A})$. Q.E.D.

We now come to our main results. In the remaining portion of this paper, W is defined as above for $L=K$, and $\mathcal{S}$ denotes the unit sphere in $B(K)$.

THEOREM 4. $\mathcal{P}=\{\lambda\mid \mathfrak{A}(\lambda)$ *has property* P$\}$ *is* μ*-measurable.*

PROOF. We shall give a measurable characterization of the set $\mathcal{P}'=\Lambda-\mathcal{P}$. We begin by considering a W-* algebra $\mathcal{B}$ on K which has property P. Given $T\in\mathcal{S}$, it follows that there are $f_k\in\mathcal{E}(\mathcal{B})$ and $B'\in\mathcal{B}'$ such that $f_k\cdot T=T_k\to B'$ weakly. Since bounded sets are weakly compact and since W determines weak convergence on bounded sets, this is equivalent to the statement that, if B_n' are dense in $\mathcal{B}_1'$ then there are B_{n_k}' such that $W(T_k-B_{n_k}')\to 0$. Next, suppose $\{U_n\}$ are dense in $U(\mathcal{B})$. Then it is easy to see that we may assume that the support of each f_k is contained in $\{U_n\}$ and that the values $f(U_n)$ are rational. This is the key to our theorem.

Let $\mathfrak{F}$ be that subset of $\mathcal{E}(\mathcal{A})$ consisting of f with support contained in the set $\{U_n\}$ and whose values $f(U_n)$ are rational. Clearly $\mathfrak{F}$ is countable. For each $f\in\mathfrak{F}$ and each pair of positive integers (k, m) define a subset $E(f, k, m)$ of $\Lambda\times\mathcal{S}$ consisting of pairs (λ, T) satisfying the following condition, where by $(f\cdot T)(\lambda)$ we mean $\sum_{U_n} f(U_n)U_n(\lambda)TU_n(\lambda)^*$.

$$W((f\cdot T)(\lambda) - A_k'(\lambda)) \geqq 1/m. \tag{1}$$

Each set $E(f, k, m)$ is closed. It follows from our remarks above that, if we let π denote the projection of $\Lambda\times\mathcal{S}$ onto Λ, then $\mathcal{P}'$ differs by a μ-null set from the set

$$E = \pi\left(\bigcup_{m=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcap_{f\in F} E(f, k, m)\right).$$

E is μ-measurable by [7, Lemma I.4.6] and the theorem follows. Q.E.D.

THEOREM 5. *$\mathcal{A}$ has property* P *iff* $\mu(P')=0$.

PROOF. Suppose $\mu(P')>0$. Then it follows from our last proof and from [7, Lemma I.4.7] that there exist an integer $m>0$, a set F of positive measure, and a μ-measurable operator-valued function T defined on F such that $(\lambda, T(\lambda))\in E(f, k, m)$ for each $f\in\mathfrak{F}$ and each k. Extend T to all of Λ by letting $T(\lambda)=0$ if $\lambda\notin F$, and set $T=\int_\Lambda \oplus T(\lambda)\mu(d\lambda)$. Then $T\in B(H)_1$ and, if $\mathcal{A}$ has property P, there are functions $g_k\in\mathcal{E}(\mathcal{A})$ and an operator $A'\in\mathcal{A}_1'$ such that $g_k\cdot T\to A'$ weakly. By [8, Lemma 1.7] and [2, Corollary III.6.13] we may assume $(g_k\cdot T\in\mathcal{Z}'$ for each $k)$ that $(g_k\cdot T)(\lambda)-A'(\lambda)\to 0$ weakly μ-a.e. In particular this must be true for μ-a.e. λ in F, which, by our remarks beginning the proof of Theorem 4, contradicts the fact that $(\lambda, T(\lambda))\in E(f, k, m)$ for each $f\in\mathfrak{F}$ and each k and all $\lambda\in F$. Hence $\mathcal{A}$ does not have property P.

To prove the converse, it suffices by Lemma 3 to show that if $\mu(P')=0$ then $\mathfrak{A}$ has property CP. We may restrict our attention to $T\in\mathcal{Z}_1'$, with $T=\int_\Lambda \oplus T(\lambda)\mu(d\lambda)$. By [7, p. 228], [8, Lemma 1.7], [2, Corollary III.6.13], and the remark following [8, Lemma 2.2] we may assume that Λ is compact and that $T(\lambda)$ is strong-* continuous in λ. Moreover, since $\mu(P')=0$, we may assume that $\mathfrak{A}(\lambda)$ has property P for every λ (see remark following [7, Corollary I.5.10]). Thus given any integer m and any $\lambda\in\Lambda$, there are $f\in\mathfrak{F}$ and A_k' such that

$$W((f\cdot T)(\lambda) - A_k'(\lambda)) < 1/m.$$

By continuity each such inequality holds on an open set, and by compactness there is a finite cover by such sets. Hence there are disjoint μ-measurable sets F_i such that $\Lambda=\bigcup_{i=1}^n F_i$ and such that for each F_i there are $f_i\in\mathfrak{F}$ and A_{k_i}' for which

$$W((f_i\cdot T)\lambda - A_{k_i}'(\lambda)) < 1/m \quad \text{for } \lambda\in F_i.$$

We now show that there are $h\in\mathcal{E}(\mathfrak{A})$ and $A'\in\mathfrak{A}'$ for which

$$W((h\cdot T)(\lambda) - A'(\lambda)) < 1/m \quad \text{for every } \lambda.$$

It then follows that $\mathfrak{A}$ has property CP, and our theorem is proved.

For convenience of notation, assume $\Lambda=F\cup G$, with f, g and A_j', A_k' as above. Let $V_1,\cdots,V_n$ be the support of f, and let $W_1,\cdots,W_m$ be the support of g. Define unitaries $U_{i,j}$ by $U_{i,j}(\lambda)=V_i(\lambda)$ if $\lambda\in F$ and $W_j(\lambda)$ if $\lambda\in G$. Define h with support on the $U_{i,j}$ by $h(U_{i,j})=f(V_i)g(W_j)$. Define $A'\in\mathfrak{A}'$ by $A'(\lambda)=A_j'(\lambda)$, $\lambda\in F$ and $A'(\lambda)=A_k'(\lambda)$, $\lambda\in G$. Then it is clear that h and A' give the desired result. Clearly the construction does not depend on the number of sets F_i, and our theorem is proved. Q.E.D.

COROLLARY 6. *If $\mathfrak{A}$ is of type* I, *then $\mathfrak{A}$ has property* P.

PROOF. It suffices to note that each type I factor on K has property P. This is clear for $B(K)$ and for finite-dimensional factors, and the general result follows, since property P is a *-isomorphism invariant [3], from the known structure of type I factors. Q.E.D.

Schwartz introduced property P in [5], [6] as a property of hyperfinite factors. Since Powers has constructed a continuum of hyperfinite type III factors [4] and Ching has constructed a continuum of type III factors not having property P [1] and therefore not hyperfinite, the following corollary has some interest.

COROLLARY 7. *Let $\mathfrak{A}$ be a W-* algebra on separable Hilbert space H. Then there is a projection $E \in \mathfrak{Z}$ such that $\mathfrak{A}_E$ has property P and such that $\mathfrak{B} = \mathfrak{A}_{I-E}$ contains no central projection $F \neq 0$ for which $\mathfrak{B}_F$ has property* P.

PROOF. Let E be the projection induced by the characteristic function of $\mathcal{P}$. Q.E.D.

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