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A CHARACTERIZATION OF REGULARITY IN TOPOLOGY

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ABSTRACT. We show in this paper that a topological space satisfies T_3 (which we do not intend to imply T_2) if and only if convergence of filters is a continuous relation. In particular, a Hausdorff space is regular if and only if convergence of filters is a continuous mapping. We propose a new, categorically motivated, definition of continuous relations between topological spaces, and we compare it with two existing continuity concepts for relations.

Let (E, τ) be a topological space. We denote by E^* the set of all filters on E which converge for τ to some point of E. For $X \subset E$, we put $X^* = \{\varphi \in E^* : X \in \varphi\}$. Then $\varnothing^* = \varnothing$ for the empty set, and

$$(X \cap Y)^* = X^* \cap Y^*, \qquad \dot{x} \in X^* \Leftrightarrow x \in X,$$

for subsets X, Y of E, $x \in E$, and $\dot{x} = \{X \subset E : x \in X\}$. We regard convergence of filters for τ as a relation $q: E^* \to E$, writing $\varphi\, q\, x$ if φ converges to x. This relation is a mapping if and only if (E, τ) is a Hausdorff space. For $X \subset E$, we have $q(X^*) = \overline{X}$, the closure of X for τ.

It seems natural to impose a topology on E^* by using the sets U^*, with U open for τ, as a basis of open sets. The preceding paragraph shows that this works, and we denote the topology of E^* thus defined by τ^*. With this notation, we state the following theorem.

THEOREM 1. *A Hausdorff space (E, τ) is regular if and only if convergence of filters on E for τ defines a continuous map $q: (E^*, \tau^*) \to (E, \tau)$.*

Instead of proving Theorem 1 directly, we generalize it. Theorem 1 is an immediate corollary of Theorem 3 below. We need some definitions.

Let $r: E \to F$ be a relation between two sets. For $X \subset E$ and $Y \subset F$, we put $y \in r(X)$ if $y \in F$ and $x\, r\, y$ for some $x \in X$, and $x \in r^{-1}(Y)$ if $x \in E$ and $x\, r\, y$ for some $y \in Y$. One sees easily that

$$r(X) \cap Y = \varnothing \Leftrightarrow X \cap r^{-1}(Y) = \varnothing.$$

If E and F are topological spaces, then r is called *upper semicontinuous*

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if $r^{-1}(Y)$ is closed in E for every closed $Y \subset F$, and r is called *lower semicontinuous* if $r^{-1}(Y)$ is open in E for every open $Y \subset F$. These concepts have been used by various authors; see e.g. [1, Chapter VI] or [3].

A relation $r: E \to F$ between topological spaces has been called continuous if r is both upper and lower semicontinuous. We propose a different definition. We call $r: E \to F$ *continuous* if, for a topological space A and mappings $f: A \to E$ and $g: A \to F$ such that $f(u)\ r\ g(u)$ for all $u \in A$, continuity of f always implies continuity of g.

This can be simplified. Let $R \subset E \times F$ be the graph of r and $f_1: R \to E$ and $g_1: R \to F$ the projections, i.e. $f_1(x, y) = x$ and $g_1(x, y) = y$ if $x\ r\ y$. Provide R with the coarsest topology for which f_1 is continuous. If r is continuous, then g_1 is continuous for this topology of R. In fact, this is not only necessary but also sufficient for continuity of r. For if $f: A \to E$ and $g: A \to F$ are mappings such that $f(u)\ r\ g(u)$ for every $u \in A$, then $f = f_1 h$ and $g = g_1 h$ for a unique mapping $h: A \to R$, and h is continuous, for the given coarse topology of R, if f is continuous. Thus continuity of f implies continuity of g if g_1 is continuous.

We shall study continuous relations elsewhere in greater detail and in a more general setting. We mention here only that all three continuity concepts defined above coincide with the usual continuity if r is a mapping, and we connect continuity with upper and lower semicontinuity by the following result.

THEOREM 2. *A continuous relation $r: E \to F$ between topological spaces is upper semicontinuous if and only if its domain $r^{-1}(F)$ is closed in E, and r is lower semicontinuous if and only if $r^{-1}(F)$ is open in E.*

PROOF. If r is upper semicontinuous, then $r^{-1}(F)$ is closed in E. Conversely, let $R \subset E \times F$ be the graph of r and $f_1: R \to E$ and $g_1: R \to F$ the projections, as above. Provide R with the coarsest topology for which f_1 is continuous, with the sets $f_1^{-1}(X)$, X closed in E, as closed sets. If r is continuous and Y closed in F, then $g_1^{-1}(Y)$ is closed in R, and thus $g_1^{-1}(Y) = f_1^{-1}(X)$ for a closed set $X \subset E$. One sees easily that $r^{-1}(Y) = X \cap r^{-1}(F)$ in this situation. Thus $r^{-1}(Y)$ is closed if $r^{-1}(F)$ is closed. The same argument, with closed sets replaced by open sets, shows that r is lower semicontinuous if and only if $r^{-1}(F)$ is open. $\square$

The following example shows that Theorem 2 has no obvious converse. For every topological space E, the full relation $r: E \to E$ with graph $E \times E$ is both upper and lower semicontinuous. On the other hand, we have $f(u)\ r\ g(u)$ for all $u \in A$ if $f: A \to E$ and $g: A \to E$ are arbitrary mappings. Thus r is continuous only if E is an indiscrete space.

We need one of the separation axioms introduced by Davis [2]. Davis calls a topological space (E, τ), with filter convergence q, an R_0 space if always $\dot{x}\ q\ y \Rightarrow \dot{y}\ q\ x$ for x, y in E. It is shown in [2] that T_1 is equivalent to the conjunction of T_0 and R_0, and that T_3 (called R_2 in [2]) always implies R_0.

THEOREM 3. *The following three statements are logically equivalent for a topological space (E, τ) with filter convergence q.*

(i) *(E, τ) is a T_3 space.*

(ii) *$q:(E^*, \tau^*) \rightarrow (E, \tau)$ is continuous.*

(iii) *(E, τ) is an R_0 space and q is upper semicontinuous.*

PROOF. Assume first T_3 and consider $f: A \rightarrow E^*$ and $g: A \rightarrow E$ with f continuous and $f(u)$ converging to $g(u)$ for all $u \in A$. If U is open in E and $g(u) \in U$, then $g(u) \in V$ and $\overline{V} \subset U$ for some open V. For this V, we have $V \in f(u)$, and $V \in f(v)$ implies $g(c) \in \overline{V}$. Thus $u \in f^{-1}(V^*)$ and $f^{-1}(V^*) \subset g^{-1}(U)$. This shows that $g^{-1}(U)$ is open, and hence g continuous.

If q is continuous, then q is upper semicontinuous by Theorem 2. If $\dot{x}\ q\ y$, let A be the space with two points u, v, and with $\{v\}$ open, but not closed. Put $f(u) = f(v) = \dot{x}$ and $g(u) = x$, $g(v) = y$. Then f is continuous, and $f(z)\ q\ g(z)$ for $z \in A$. Thus g is continuous. If V is open and $x \in V$, then $g^{-1}(V)$ is open and $u \in g^{-1}(V)$. Thus $g^{-1}(V) = A$, and $y \in V$. This shows that also $\dot{y}\ q\ x$, and E is R_0.

Assume now (iii), and let F be closed in E and $x \in E \backslash F$. If $\dot{x}\ q\ y$, then $\dot{y}\ q\ x$, and $y \in F$ would imply $x \in \overline{F} = F$. Thus $\dot{x} \notin q^{-1}(F)$. It follows that $\dot{x} \in V^*$ for an open set V with $V^* \cap q^{-1}(F) = \varnothing$. But then $x \in V$, and $\overline{V} \cap F = q(V^*) \cap F = \varnothing$. Thus E satisfies T_3. □

The following example shows that R_0 cannot be omitted from Theorem 3. The space with two points and three open sets (used in the proof of the theorem) is T_0 but not T_1, and hence a fortiori not T_3 or R_0. But one sees easily that q is upper semicontinuous for this space.

REMARK. All results of this note remain valid if E^* is replaced by a set of convergent filters which contains all convergent ultrafilters.

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