PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 3, August 1971

THE CENTER OF THE FREE PRODUCT OF DISTRIBUTIVE LATTICES[1]

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ABSTRACT. The object of this paper is to show that for members L and M in the class of distributive lattices with zero and unit, the center of the free product of L and M is the free product of their centers.

We can assume that L and M are subalgebras (that is (0, 1)-sublattices) of their free product, which we denote by $L*M$. This means [2] that $L\cup M$ generates $L*M$ and for $a_1, a_2\in L$, $b_1, b_2\in M$:

$$(1)\qquad a_1b_1 \leqq a_2 + b_2 \quad \text{implies} \quad a_1 \leqq a_2 \quad \text{or} \quad b_1 \leqq b_2.$$

Since the center $C(L)$ of L is exactly its subalgebra of complemented elements [1, p. 67], we have immediately that $C(L)*C(M)\subseteq L*M$. But an element $x=\sum_{i=1}^{p} a_ib_i$, where $p\geqq 1$, $a_i\in C(L)$, $b_i\in C(M)$, has a complement, namely $\prod_{i=1}^{p} (a_i'+b_i')$, where a_i' is the complement of a_i in L and b_i' is the complement of b_i in M. So $C(L)*C(M)\subseteq C(L*M)$.

To prove the reverse inclusion, it suffices to show that if $x\in C(L*M)$ and

$$(2)\qquad x = \sum_{i=1}^{p} a_ib_i, \qquad p \geqq 1, \quad a_ib_i \neq 0, \quad a_i \in L, \quad b_i \in M,$$

then any a_i that appears in (2) can be replaced by a member of $C(L)$ and still leave (2) valid. Indeed, if this replacement is possible then each a_i can be successively replaced by members of $C(L)$ and then the whole process repeated for each b_i; thus showing that $x\in C(L)*C(M)$.

Now to prove that this replacement is possible suppose $x\neq 0, 1$ has the representation in (2) and that x' is the complement of x in $C(L*M)$. So

$$x' = \sum_{j=1}^{q} \alpha_j\beta_j \quad \text{for some } q \geqq 1,\ \alpha_j \in L, \beta_j \in M.$$

Received by the editors October 5, 1970.

AMS 1969 *subject classifications*. Primary 0635, 0640; Secondary 0805, 0810.

Key words and phrases. Free product, center, complemented elements, zero, unit.

[1] The research and preparation of this paper was supported in part by NSF GP 11893.

We will replace a_1 by an element of $C(L)$. If $a_1 \in C(L)$ we are finished so suppose $a_1 \notin C(L)$. Noting the convention $\sum \varnothing = 0$ and the fact that (1) implies that for each $j \in \{1, \cdots, q\}$ either $a_1\alpha_j = 0$ or $b_1\beta_j = 0$, we have:

$$\begin{aligned}\left(a_1 + \sum_{i=2}^{p} b_i\right) + \left(\sum \{\alpha_i \mid a_1\alpha_j = 0\} + \sum\{\beta_j \mid b_1\beta_j = 0\}\right)&\\ \geqq x + x' = 1.&\end{aligned} \tag{3}$$

Now $a_1 \sum \{\alpha_j \mid a_1\alpha_j = 0\} = 0$ so $a_1 + \sum \{\alpha_j \mid a_1\alpha_j = 0\} \neq 1$. Applying (1) to (3) we obtain $\sum_{i=2}^{p} b_i + \sum \{\beta_j \mid b_1\beta_j = 0\} = 1$. This implies

$$b_1 \leqq \sum_{i=2}^{p} b_i. \tag{4}$$

Let $\{S_1, \cdots, S_r\}$ be all subsets S of $\{b_2, \cdots, b_r\}$ with the property that $b_1 \leqq \sum (S)$. Note that from (4) and (2), $r \geqq 1$ and $S_j \neq \varnothing$, for $j = 1, \cdots, r$. Set $T_j = \{a_k \mid b_k \in S_j\}$ for $j = 1, \cdots, r$. We will show that the required replacement for a_1 is $A = a_1 + \prod (T_1) + \cdots + \prod (T_r)$. Clearly $x \leqq Ab_1 + a_2b_2 + \cdots + a_pb_p$. On the other hand, for each $j \in \{1, \cdots, r\}$, $\prod (T_j)b_1 \leqq \prod (T_j) \sum (S_j) \leqq \sum \{a_kb_k \mid b_k \in S_j\} \leqq x$. So $Ab_1 \leqq x$ and

$$x = Ab_1 + a_2b_2 + \cdots + a_pb_p. \tag{5}$$

It remains to prove $A \in C(L)$. Let $\mathfrak{T}$ be the family of all sets T which consist of exactly one member from each of the sets $T_1, \cdots, T_r$. For such a $T \in \mathfrak{T}$, (5) yields

$$\begin{aligned}&\left(A + \sum (T) + \sum \{b_i \mid a_i \notin T, i \geqq 2\}\right)\\ &\quad + \left(\sum \{\alpha_j \mid A\alpha_j = 0\} + \sum \{\beta_j \mid b_1\beta_j = 0\}\right) \geqq x + x' = 1.\end{aligned} \tag{6}$$

Now if $\sum \{b_i \mid a_i \notin T, i \geqq 2\} + \sum \{\beta_j \mid b_1\beta_j\} = 1$ then

$$b_1 \leqq \sum \{b_i \mid a_i \notin T, i \geqq 2\} \quad \text{so} \quad \{b_i \mid a_i \notin T, i \geqq 2\} = S_j$$

for some $j \in \{1, \cdots, r\}$. But this is impossible, for by the definition of T, there is a member $a_{i_0} \in T_j \cap T$, where $i_0 \geqq 2$, so

$$b_{i_0} \in S_j - \{b_i \mid a_i \notin T, i \geqq 2\} = \varnothing.$$

By applying (1) to (6) we obtain for each $T \in \mathfrak{T}$:

$$A + \sum (T) + \sum \{\alpha_j \mid A\alpha_j = 0\} = 1.$$

Thus,

$$\begin{aligned}1 &= A + \prod\{\textstyle\sum(T) \mid T \in \mathfrak{z}\} + \sum\{\alpha_j \mid A\alpha_j = 0\}\\ &= A + (\textstyle\prod(T_1) + \cdots + \prod(T_r)) + \sum\{\alpha_j \mid A\alpha_j = 0\}\\ &= A + \textstyle\sum\{\alpha_j \mid A\alpha_j = 0\}.\end{aligned}$$

This means that A and $\sum\{\alpha_j | A\alpha_j = 0\}$ are complements in L, which completes the proof.

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