PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 3, August 1971

UNIQUENESS OF SOLUTIONS OF CERTAIN BOUNDARY VALUE PROBLEMS FOR ULTRAHYPERBOLIC EQUATIONS

J. B. DIAZ AND EUTIQUIO C. YOUNG

ABSTRACT. The paper gives necessary and sufficient conditions for the uniqueness of solutions of the Dirichlet and Neumann problems for ultrahyperbolic partial differential equations in cylindrical domains.

1. Introduction. In [1], Owens has given sufficient conditions for uniqueness of solutions of the ultrahyperbolic differential equation

$$\sum_{i=1}^{m} u_{x_i x_i} - \sum_{j=1}^{n} u_{y_j y_j} = 0 \qquad (m \geqq 2,\ n \geqq 2) \tag{1}$$

for certain mixed problems having elliptic and hyperbolic nature. However, problems of the Dirichlet and Neumann type are not considered. The purpose of this note is to present necessary and sufficient conditions for the uniqueness of solutions of the Dirichlet and Neumann problems for the more general ultrahyperbolic differential equation

$$Lu \equiv \sum_{i=1}^{m} u_{x_i x_i} - \sum_{j,k=1}^{n} (a_{jk} u_{y_j})_{y_k} + cu = 0 \tag{2}$$

in a domain $Q = X \times Y$, where X is a hyper-parallelepiped defined by $0 < x_i < a_i$, $1 \leqq i \leqq m$, and Y is a bounded domain in the n-dimensional space $y_1, \cdots, y_n$. The procedure we shall use here is an extension of that employed in [2], [3], [4], and [5] for the normal hyperbolic equations. Since our treatment remains valid for $m \geqq 1$, our results here, therefore, contain also those obtained in [2], [3], and [4].

Throughout this paper, we assume that the coefficients a_{jk} and c depend only on the variables $y_1, \cdots, y_n$, and are continuous functions of these variables with $c \geqq 0$ in Y. Moreover, we assume that the coefficient matrix (a_{jk}) is symmetric and positive definite, and that a_{jk}, together with the boundary ∂Y of Y, are sufficiently smooth in order to ensure the existence of complete sets of eigenfunctions for the eigenvalue problems that we will need below.

Received by the editors November 9, 1970.

AMS 1970 *subject classifications*. Primary 35L10, 35L20; Secondary 35A05.

Key words and phrases. Uniqueness of solution, ultrahyperbolic equations, Dirichlet and Neumann problems, mixed boundary value problems, eigenvalues, eigenfunctions, complete set of eigenfunctions.

For brevity, we shall let x and y stand for the sets of variables $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_n)$, respectively, and write $u(x, y)$ for $u(x_1, \cdots, x_m, y_1, \cdots, y_n)$.

2. **The Dirichlet problem.** It suffices to consider the homogeneous problem

$$Lu = 0 \quad \text{in } Q, \qquad u = 0 \quad \text{on } \partial Q. \tag{3}$$

THEOREM 1. *Let λ_k $(k=1, 2, \cdots)$ be the eigenvalues of the problem*

$$\sum_{j,k=1}^{n} (a_{jk}v_{y_j})_{y_k} - cv + \lambda v = 0 \quad \text{in } Y, \qquad v = 0 \quad \text{on } \partial Y. \tag{4}$$

Then every solution $u \in C^2(Q) \cap C^1(\bar{Q})$ of the problem (3) *vanishes identically in Q if, and only if,*

$$\pi^2 \sum_{i=1}^{m} (k_i/a_i)^2 \neq \lambda_k \tag{5}$$

for all nonzero integers $k_1, \cdots, k_m$.

PROOF. The necessity part follows easily. Indeed, suppose there exist nonzero integers $p_1, \cdots, p_m$, and an eigenvalue λ_p of (4) such that

$$\pi^2 \sum_{i=1}^{m} (p_i/a_i)^2 = \lambda_p.$$

Let v_p be an eigenfunction of (4) corresponding to λ_p. Then the function

$$u(x, y) = \prod_{i=1}^{m} \sin\left(\frac{p_i \pi}{a_i} x_i\right) v_p(y) \tag{6}$$

is a nontrivial solution of the problem (3), as is readily verified.

Conversely, suppose that condition (5) holds. By the divergence theorem, we have

$$\begin{aligned} &\int\int_Q (uLw - wLu)\,dx\,dy \\ &\quad = \int_{\partial Q} \left[\sum_{i=1}^{m} (uw_{x_i} - u_{x_i}w)\nu_{x_i} - \sum_{j,k=1}^{n} a_{jk}(uw_{y_j} - u_{y_j}w)\nu_{y_k} \right] dS \end{aligned} \tag{7}$$

where $dx = dx_1 \cdots dx_m$, $dy = dy_1 \cdots dy_n$, dS is the surface element on ∂Q, and ν_{x_i} and ν_{y_k} are the direction cosines of the outward normal vector on ∂X and ∂Y respectively. Now let u be a solution of (3) and choose

$$w(x, y) = \prod_{i=1}^{m-1}\left(\sin\frac{k_i\pi}{a_i}x_i\right)(\sin\mu_m x_m)v_k(y) \tag{8}$$

where the nonzero integers $k_1, \cdots, k_{m-1}$ and the nonzero constant μ_m satisfy the relation

$$\pi^2\sum_{i=1}^{m-1}(k_i/a_i)^2 + \mu_m^2 = \lambda_k \tag{9}$$

with v_k being an eigenfunction of (4) corresponding to the eigenvalue λ_k. Notice that μ_m may very well be an imaginary number. Then, as is readily seen, the function (8) satisfies $Lw=0$ and vanishes on ∂Q, except on $x_m=a_m$ in view of condition (5). Substitution of these functions in the formula (7) thus yields

$$\int_Y\left\{\int_{X'} u_{x_m}(x', a_m, y)\prod_{i=1}^{m-1}\left(\sin\frac{k_i\pi}{a_i}x_i\right)dx'\right\}v_k(y)dy = 0 \tag{10}$$

for all eigenfunctions v_k and for all nonzero integers $k_1, \cdots, k_{m-1}$. Here $x'=(x_1, \cdots, x_{m-1})$, $dx'=dx_1\cdots dx_{m-1}$, and X' denotes the subspace $x_1, \cdots, x_{m-1}$ of X. Since the set of eigenfunctions $\{v_k\}$ is complete, equation (10) states that

$$\int_{X'} u_{x_m}(x', a_m, y)\prod_{i=1}^{m-1}\left(\sin\frac{k_i\pi}{a_i}x_i\right)dx' = 0.$$

This, in turn, implies that

$$u_{x_m}(x', a_m, y) = 0 \tag{11}$$

in view of the completeness of the set of eigenfunctions

$$\left\{\prod_{i=1}^{m-1}\sin\frac{k_i\pi}{a_i}x_i\right\} \quad \text{in } X'.$$

Now let us integrate the differential identity

$$\begin{aligned}(2x_m u_{x_m} + u)Lu &= -2u_{x_m}^2 + \left[x_m\left(\sum_{j,k=1}^{n} a_{jk}u_{y_j}u_{y_k} - \sum_{i=1}^{m} u_{x_i}^2 + cu^2\right)\right]_{x_m} \\ &\quad + \sum_{i=1}^{m}\left[(2x_m u_{x_m} + u)u_{x_i}\right]_{x_i} - \sum_{j,k=1}^{n}\left[a_{jk}(2x_m u_{x_m} + u)u_{y_j}\right]_{y_k}\end{aligned} \tag{12}$$

over Q and use the divergence theorem. Since $u=0$ on ∂Q and $u_{x_m}=0$

on $x_m = a_m$, all the surface integrals resulting from (12) vanish and we have

$$\int\int_Q (2x_m u_{x_m} + u) Lu\, dx\, dy = -2 \int\int_Q u_{x_m}^2\, dx\, dy. \tag{13}$$

Since $Lu = 0$, this implies that u is independent of x_m. But $u = 0$ on $x_m = a_m$, hence $u \equiv 0$ in Q.

Our proof of Theorem 1 shows that, whenever condition (5) holds, it is not necessary to prescribe the values of the normal derivative of u on a face of X, as required in Theorem 2 of [1].

3. **The Neumann problem.** Consider next the homogeneous Neumann problem

$$Lu = 0 \quad \text{in } Q, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial Q; \tag{14}$$

where on $X \times \partial Y$, the derivative $\partial u/\partial n$ is defined by

$$\frac{\partial u}{\partial n} = \sum_{j,k=1}^{n} a_{jk} u_{y_j} \nu_{y_k}.$$

When the quantities λ_k are taken to be eigenvalues of a corresponding boundary value problem of the Neumann type, it turns out that condition (5) is also necessary and sufficient for $u = 0$ to be the only solution of the problem (14), provided $c \not\equiv 0$.

THEOREM 2. *Let λ_k be the nonzero eigenvalues of the problem*

$$\sum_{j,k=1}^{n} (a_{jk} v_{y_j})_{y_k} - cv + \lambda v = 0 \quad in\ Y, \qquad \frac{\partial v}{\partial n} = 0 \quad on\ \partial Y. \tag{15}$$

Then every solution $u \in C^2(Q) \cap C^1(\overline{Q})$ of (14) vanishes identically (or $u = \text{const}$ in case $c \equiv 0$) if, and only if,

$$\pi^2 \sum_{i=1}^{m} (k_i/a_i)^2 \neq \lambda_k \tag{16}$$

for all integers $k_1, \cdots, k_m$.

PROOF. If there exist integers $p_1, \cdots, p_m$, and a nonzero eigenvalue λ_p of (15), for which (16) does not hold, then a nonconstant and hence a nontrivial solution of (14) is given by

$$u(x, y) = \prod_{i=1}^{m} \left(\cos \frac{k_i \pi}{a_i} x_i \right) v_p(y)$$

where v_p is an eigenfunction corresponding to λ_p.

On the other hand, suppose that condition (16) holds. Let u be a solution of (14) and choose

$$w(x, y) = \prod_{i=1}^{m-1}\left(\cos \frac{k_i\pi}{a_i} x_i\right)(\cos \mu_m x_m)v_k(y) \tag{17}$$

where the integers $k_1, \cdots, k_{m-1}$ and the constant μ_m satisfy the relation (9). Substituting these functions in the formula (7), and using condition (16), we then obtain

$$\int_Y \left\{\int_{X'} u(x', a_m, y) \prod_{i=1}^{m-1}\left(\cos \frac{k_i\pi}{a_i} x_i\right) dx'\right\} v_k(y)dy = 0 \tag{18}$$

for all integers $k_1, \cdots, k_{m-1}$, and for all eigenfunctions v_k corresponding to the nonzero eigenvalues λ_k. By the completeness of the set $\{v_k\}$, equation (18) implies that

$$\int_{X'} u(x', a_m, y) \prod_{i=1}^{m-1}\left(\cos \frac{k_i\pi}{a_i} x_i\right) dx' = \nu = \text{const.} \tag{19}$$

with $\nu \equiv 0$ in case $c > 0$. Notice that when $c \equiv 0$, $v_0 = 1$ is an eigenfunction of (15) corresponding to the eigenvalue $\lambda_0 = 0$. Now, since the set of eigenfunctions $\{\prod_{i=1}^{m-1} \cos (k_i\pi/a_i)x_i\}$ is complete in X', equation (19) in turn implies that $u(x', a_m, y) = \text{const.}$, which is zero if $c > 0$.

Let us consider the case $c > 0$. By integrating the identity (12) over Q and using the fact $u = 0$ on $x_m = a_m$, we again arrive at equation (13) from which the conclusion that $u \equiv 0$ in Q follows.

In case $c \equiv 0$, the above argument yields the result $u = \text{const.}$ in Q.

4. **Mixed boundary value problems.** The method we have used to establish Theorems 1 and 2 above can also be employed to obtain uniqueness results for equation (1) with mixed boundary conditions of the following type:

$$u = 0 \quad \text{on } \partial X \times \overline{Y}, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \overline{X} \times \partial Y; \tag{20}$$

$$\begin{aligned} u &= 0 \text{ on } x_i = 0, \ x_i = a_i \qquad (1 \leqq i \leqq p), \\ u_{x_j} &= 0 \text{ on } x_j = 0, \ x_j = a_j \qquad (p + 1 \leqq j \leqq m), \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \overline{X} \times \partial Y; \end{aligned} \tag{21}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial X \times \overline{Y}, \qquad u = 0 \text{ on } \overline{X} \times \partial Y; \tag{22}$$

$$(23)\qquad \begin{aligned} u &= 0 \text{ on } x_i = 0,\ x_i = a_i && (1 \leqq i \leqq p), \\ u_{x_j} &= 0 \text{ on } x_j = 0,\ x_j = a_j && (p+1 \leqq j \leqq m), \\ u &= 0 \text{ on } \overline{X} \times \partial Y. \end{aligned}$$

Specifically, we have the following uniqueness theorems corresponding to each of these boundary conditions.

THEOREM 3. *Every solution* $u \in C^2(Q) \cap C^1(\overline{Q})$ *of equation* (2) *in* Q *satisfying the boundary conditions* (20) [*or* (21)], *vanishes identically in* Q *if, and only if,* (5) *holds for all nonzero integers* $k_1, \cdots, k_m$ [*or nonzero integers* $k_1, \cdots, k_p$ *and integers* $k_{p+1}, \cdots, k_m$], *where* λ_k $(k=1, \cdots, m)$ *are the eigenvalues of the problem* (15).

THEOREM 4. *Every solution* $u \in C^2(Q) \cap C^1(\overline{Q})$ *of equation* (2) *in* Q *satisfying the boundary conditions* (22) [*or* (23)], *vanishes identically in* Q *if, and only if,* (5) *holds for all integers* $k_1, \cdots, k_m$ [*or nonzero integers* $k_1, \cdots, k_p$ *and integers* $k_{p+1}, \cdots, k_m$], *where* λ_k *are the eigenvalues of the problem* (4).

The second author wishes to acknowledge, with gratitude, support from the Mathematics Department of the Florida State University and the Albert Einstein Chair, held by Dr. J. B. Diaz, of the Rensselaer Polytechnic Institute at which this joint work was carried out while the second author was on leave.

REFERENCES

1. O. G. Owens, *Uniqueness of solutions of ultrahyperbolic partial differential equations*, Amer. J. Math. **69** (1947), 184–188. MR **8**, 466.

2. D. R. Dunninger and E. C. Zachmanoglou, *The condition for uniqueness of solutions of the Dirichlet problem for the wave equation in coordinate rectangles*, J. Math. Anal. Appl. **20** (1967), 17–21. MR **37** #1807.

3. ———, *The condition for uniqueness of the Dirichlet problem for hyperbolic equations in cylindrical domains*, J. Math. Mech. **18** (1969), 763–766. MR **38** #4817.

4. V. G. Sigillito, *On the uniqueness of solutions of certain improperly posed problems*, Proc. Amer. Math. Soc. **24** (1970), 828–831. MR **40** #7612.

5. E. C. Young, *Uniqueness of solutions of the Dirichlet and Neumann problems for hyperbolic equations*, Trans. Amer. Math. Soc. (to appear).

RENSSELAER POLYTECHNIC INSTITUTE, TROY, NEW YORK 12181

FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306