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EXISTENCE AND CONTINUOUS DEPENDENCE FOR A CLASS OF NONLINEAR NEUTRAL-DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper presents existence, uniqueness, and continuous dependence theorems for solutions of initial-value problems for neutral-differential equations of the form

$$x'(t) = f(t, x(t), x(g(t, x)), x'(h(t, x))), \qquad x(0) = x_0,$$

where f, g, and h are continuous functions with $g(0, x_0) = h(0, x_0) = 0$. The existence of a continuous solution of the functional equation $z(t) = f(t, z(h(t)))$ is proved as a corollary.

1. Introduction. In this paper we consider the initial-value problem (IVP) for the functional-differential equation of neutral type

$$x'(t) = f(t, x(t), x(g(t, x(t))), x'(h(t, x(t)))), \tag{1}$$

with the initial condition

(2a) $x(0) = x_0$.

Here $f(t, x, y, z)$, $g(t, x)$ and $h(t, x)$ are continuous functions with $g(0, x_0) = h(0, x_0) = 0$. We assume further that the algebraic equation $z = f(0, x_0, x_0, z)$ has a real root z_0, and we require that

(2b) $x'(0) = z_0$.

Existence and uniqueness theorems for IVP's for equation (1) have been proved by R. D. Driver [1] for the case where $h(t, x) < t$, and recently by J. K. Hale and M. A. Cruz [3] provided that f is linear in the argument $x'(h(t, x))$. We prove an existence theorem without these hypotheses, and a uniqueness theorem in case h is independent of x. Hale and Cruz [3] have also obtained continuity theorems for the quasilinear case mentioned above, while Driver [2] has proved a continuity theorem for IVP's for equations of the form (1) in case g and h are both independent of x, and $h(t) < t$ for all t. We obtain here a continuous dependence result for the IVP (1)–(2a)–(2b) provided that the function h is independent of x. Finally we obtain a result on existence of continuous solutions of certain nonlinear functional equations as a corollary of our existence and uniqueness theorems.

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2. **Existence.** Let $\alpha>0$, and let $J=[-\alpha, \alpha]$. We shall make the following assumptions concerning the IVP (1)–(2a)–(2b):

(i) $f(t, x, y, z)$ is continuous in some region in R^4 containing

$$P = \{(t, x, y, z): |t| \leqq \alpha,\ |x - x_0| \leqq \beta,\ |y - x_0| \leqq \beta,\ |z| \leqq M\}$$

where α, β and $M>|z_0|$ are positive constants. We assume that $\alpha \leqq \beta/M$ and that $\sup_{(t,x,y,z)\in P} |f(t, x, y, z)| < M$.

(ii) $g(t, x)$ and $h(t, x)$ are continuous in the projection $\tilde{R}$ of P in the (t, x) space; g and h both map $\tilde{R}$ into J, with $g(0, x_0)=h(0, x_0)=0$, and $h(t, x)$ satisfies the Lipschitz conditions

$$|h(t_1, x_1) - h(t_2, x_2)| \leqq k_1|t_1 - t_2| + k_2|x_1 - x_2|$$

for all (t_1, x_1), $(t_2, x_2)\in R$, where k_1 and k_2 are nonnegative constants with $k_1+k_2M \leqq 1$.

(iii) The function $f(t, x, y, z)$ satisfies the Lipschitz condition

$$|f(t, x, y, z_1) - f(t, x, y, z_2)| \leqq L_z|z_1 - z_2|$$

for all (t, x, y, z_1), $(t, x, y, z_2)\in P$, where $L_z<1$.

We shall prove the following theorem:

THEOREM 1. *Under the hypotheses* (i)–(iii), *the IVP* (1)–(2a)–(2b) *has at least one solution which is continuously differentiable on J.*

PROOF. Let X be the Banach space of continuous functions on J with uniform norm. Let

$$S = \{z \in X : z(0) = z_0,\ \|z\| \leqq M\}.$$

Define the mapping $T: S\to S$ as follows: for $z\in S$, let

$$Tz(t) = f(t, I(z, t), I(z, g(t, I(z, t))), z(h(t, I(z, t)))),$$

where

$$I(z, t) = x_0 + \int_0^t z(s)ds.$$

It is easy to verify that T is a continuous map of S into S. By continuity of f, if $z\in S$ and $t\in J$, for each $\epsilon>0$, there exists $\delta(\epsilon)>0$ such that if t_1 and $t_2\in J$, and $|t_1-t_2|<\delta(\epsilon)$, then

$$|f(t_1, I(z, t_1), I(z, g(t_1, I(z, t_1))), z(h(t, I(z, t)))) \\ - f(t_2, I(z, t_2), I(z, g(t_2, I(z, t_2))), z(h(t, I(z, t))))| < \epsilon.$$

Let

$$S_\epsilon = \{z \in S: |z(t_1) - z(t_2)| \leqq \epsilon/(1 - L_z) \quad \text{for all } t_1, t_2 \in J, |t_1 - t_2| \leqq \delta(\epsilon)\}.$$

If $z \in S_\epsilon$, and if $t_1, t_2 \in J$ with $|t_1 - t_2| \leqq \delta(\epsilon)$, then

$$|Tz(t_1) - Tz(t_2)| \leqq \epsilon + \epsilon L_z/(1 - L_z) = \epsilon/(1 - L_z).$$

Thus $TS_\epsilon \subset S_\epsilon$. We note that S_ϵ is closed, bounded and convex. Let $S_0 = \bigcap_{\text{all } \epsilon > 0} S_\epsilon$. S_0 is nonempty, closed, bounded and convex, and by the Ascoli-Arzela theorem, S_0 is compact. Since $TS_\epsilon \subset S_\epsilon$ for all $\epsilon > 0$, $TS_0 \subset S_0$. Hence by the Schauder theorem, T has at least one fixed point $z(t)$. Integration yields the required solution of (1)–(2a)–(2b).

3. **Uniqueness.** In case $h(t, x)$ is independent of x, we obtain the following uniqueness result:

THEOREM 2. *In addition to the hypotheses of Theorem* 1, *suppose that:*
(iv) $h(t, x) \equiv h(t)$ *is independent of* x.
(v) f *and* g *satisfy the Lipschitz conditions*

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leqq L\{|x_1 - x_2| + |y_1 - y_2|\} + L_z|z_1 - z_2|$$

with $L_z < 1$;

$$|g(t, x_1) - g(t, x_2)| \leqq L_g|x_1 - x_2|,$$

uniformly in their domains.

Then there exists γ_0, $0 < \gamma_0 \leqq \alpha$, *such that there is a unique continuously differentiable solution of the IVP* (2)–(3a)–(3b) *on* $[-\gamma_0, \gamma_0]$.

PROOF. Under the hypotheses of the theorem, if $z \in S$, $0 < \gamma \leqq \alpha$, and $t \in [-\gamma, \gamma]$,

$$\begin{aligned} |Tz_1(t) - Tz_2(t)| &\leqq L\{|I(z_1, t) - I(z_2, t)| \\ &\qquad + |I(z_1, g(t, I(z_1, t))) - I(z_2, g(t, I(z_2, t)))|\} \\ &\quad + L_z|z_1(h(t)) - z_2(h(t))| \\ &\leqq L\gamma\|z_1 - z_2\| + L\gamma\|z_1 - z_2\| \\ &\quad + LL_gM\gamma\|z_1 - z_2\| + L_z\|z_1 - z_2\| \\ &= [\gamma L(2 + ML_g) + L_z]\|z_1 - z_2\|. \end{aligned}$$

Hence if γ is sufficiently small, the mapping T is a contraction, and the statement of the theorem follows by integration.

REMARK. A uniqueness theorem will follow also from the theorem in the next section.

4. **Continuous dependence.** For $i=1, 2$, consider the IVP's
(1.i) $x_i'(t)=f_i(t, x_i(t), x_i(g_i(t, x_i(t))), x_i'(h_i(t)))$,
(2.ia) $x_i(0)=x_{i0}$,
(2.ib) $x_i'(0)=z_{i0}$,
under hypotheses analogous to (i)–(v):

(H1) For $i=1, 2, f_i(t, x, y, z)$ is continuous in some domain $D\subset R^4$ containing both of the sets

$$P_i = \{(t, x, y, z): |t| \leqq a, |x - x_{i0}| \leqq b, |y - x_{i0}| \leqq b, |z| \leqq M\},$$

where x_{i0} are constants, a, b, and $M>|z_{i0}|$ are constants with $\sup_{(t,x,y,z)\in D} |f_i(t, x, y, z)| < M$, and z_{i0} is a real root of the equation $z=f_i(t, x_{i0}, x_{i0}, z)$.

(H2) For $i=1, 2$, $g_i(t, x)$ is continuous in the projection of D in the (t, x) plane, and $h_i(t)$ is continuous on $[-a, a]$, with $|g_i(t, x)| \leqq |t|$; $|h_i(t)| \leqq |t|$.

(H3) The functions f_1 and g_1 satisfy the conditions satisfied by f and g respectively in §3.

THEOREM 3. *Let* (H1)−(H3) *be satisfied, let* $\alpha=\min(a, b/M)$ *and suppose that for* $i=1, 2$, $x_i(t)$ *is a continuously differentiable function which satisfies* (1.i)−(2.ia)−(2.ib), *with*

$$|x_{10} - x_{20}| = \epsilon_0 < \alpha M,$$

and there exist nonnegative constants $\epsilon_f, \epsilon_g, \epsilon_h$ *such that*

$$|f_1(t, x, y, z) - f_2(t, x, y, z)| \leqq \epsilon_f,$$
$$|g_1(t, x) - g_2(t, x)| \leqq \epsilon_g,$$
$$|h_1(t) - h_2(t)| \leqq \epsilon_h$$

in their respective domains. Then if ϵ_h *is sufficiently small, for all* $t\in[-\alpha, \alpha]$,

$$|x_1(t) - x_2(t)| \leqq \epsilon_0 + C_{\epsilon,x_1}\left[\exp\left(\frac{(2 + ML_g)L|t|}{1 - L_z}\right) - 1\right] \tag{3}$$

where

$$C_{\epsilon,x_1} = \frac{\epsilon_f + (2 + ML_g)\epsilon_0 + ML\epsilon_g + L_z\epsilon_{x_1,h}}{L(2 + ML_g)}$$

and for each fixed solution $x_1(t)$, *the quantity* $\epsilon_{x_1,h}$ *tends to zero as* $\epsilon_h\to 0$.

PROOF. Let $\eta > 0$. By continuity of $x_1'(t)$, there exists $\delta > 0$ such that if $t, \tau \in [0, \alpha]$ and $|t-\tau| < \delta$, then $|x_1'(t) - x_1'(t)| < \eta$. We suppose that $\epsilon_h < \delta$. Set $z_i(t) = x_i'(t)$, $i = 1, 2$. The functions z_i satisfy the equations

$$\text{(4.i)} \quad \begin{aligned} z_i(t) = f_i\Bigg(t,\ & x_{i0} + \int_0^t z_i(s)ds, \\ & x_{i0} + \int_0^{g_i\left(t, x_{i0} + \int_0^t z_i(\sigma)d\sigma\right)} z_i(s)ds,\ z_i(h_i(t))\Bigg). \end{aligned}$$

Using the Lipschitz continuity of f_1, and the definitions of the quantities ϵ_0, ϵ_f, ϵ_g and η, we obtain from (4.1) and (4.2) the estimate

$$\begin{aligned} |z_1(t) - z_2(t)| \leqq \epsilon_f &+ L\left\{\epsilon_0 + \left|\int_0^t |z_1(s) - z_2(s)|\, ds\right|\right\} \\ &+ L\left\{\epsilon_0 + \left|\int_0^{g_2\left(t, x_{20} + \int_0^t z_2(\sigma)d\sigma\right)} |z_1(s) - z_2(s)|\, ds\right|\right. \\ &+ \left|\int_{g_2\left(t, x_{20} + \int_0^t z_2(\sigma)d\sigma\right)}^{g_1\left(t, x_{20} + \int_0^t z_2(\sigma)d\sigma\right)} |z_1(s)|\, ds\right| \\ &\left.+ \left|\int_{g_1\left(t, x_{20} + \int_0^t z_2(\sigma)d\sigma\right)}^{g_1\left(t, x_{10} + \int_0^t z_1(\sigma)d\sigma\right)} |z_1(s)|\, ds\right|\right\} \\ &+ L_z|z_1(h_2(t)) - z_2(h_2(t))| + L_z\eta. \end{aligned}$$

The *a priori* bound on $z_1(t)$ and the Lipschitz condition on $g_1(t, x)$, together with the fact that $|g_2(t, x)| \leqq |t|$, yield

$$\begin{aligned} |z_1(t) - z_2(t)| \leqq \epsilon_f &+ (2 + ML_g)L\epsilon_0 + ML\epsilon_g + L_z\eta \\ &+ (1 + ML_g)L\left|\int_0^t |z_1(s) - z_2(s)|\, ds\right| \\ &+ L\max\left\{\left|\int_0^t |z_1(s) - z_2(s)|\, ds\right|, \left|\int_{-t}^0 |z_1(s) - z_2(s)|\, ds\right|\right\} \\ &+ L_z|z_1(h_2(t)) - z_2(h_2(t))|. \end{aligned}$$

Let $K=\epsilon_f+(2+ML_g)L\epsilon_0+ML\epsilon_g+L_z\eta$, and

$$R(t) = \max_{|s|\leqq|t|} |z_1(s) - z_2(s)|.$$

Then, on $[0,\alpha]$ we have

$$R(t) \leqq K + (2 + ML_g)L\int_0^t R(s)ds + L_zR(h_2(t)),$$

and since R is an even function, is nondecreasing, and $|h_2(t)| \leqq |t|$,

$$R(t) \leqq \frac{K}{1 - L_z} + \frac{(2 + ML_g)L}{1 - L_z}\int_0^t R(s)ds.$$

By the Gronwall inequality

$$R(t) \leqq \frac{K}{1 - L_z}\exp\left(\frac{(2 + ML_g)Lt}{1 - L_z}\right) \tag{5}$$

and integration leads to

$$\begin{aligned} |x_1(t) - x_2(t)| &\leqq \epsilon_0 + \int_0^t R(s)ds \\ &\leqq \epsilon_0 + \frac{K}{(2 + ML_g)L}\left[\exp\left(\frac{(2 + ML_g)Lt}{1 - L_z}\right) - 1\right], \end{aligned}$$

and setting $C_{\epsilon,x_1}=K/(2+ML_g)L$, we obtain (3) on $[0, \alpha]$. Since R is an even function, the estimate (5) holds on $[-\alpha, 0]$ if t is replaced by $-t$. Thus analogously the estimate (3) holds also on $[-\alpha, 0]$ and the proof is complete.

5. **Nonlinear functional equations.** As a corollary to our existence and uniqueness results, we note that if $f(t, x, y, z)$ is independent of x and y, and $h(t, x)$ is independent of x, the problem (1)–(2b) has the form of the functional equation

(5) $z(t)=f(t, z(h(t)))$,

(6) $z(0)=z_0$,

where z_0 is a root of $z=f(0, z)$. Theorems 1 and 2 then yield at once:

THEOREM 4. *Let $f(t, z)$ be continuous in some region in R^2 containing $P_1=\{t: |t| \leqq\alpha,\ |z| \leqq M\}$, where α and M are positive constants such that $\sup_{(t,z)\in P_1} |f(t, z)| < M$, and $M>|z_0|$ where z_0 is a real root of $z=f(0, z)$. Let f satisfy the Lipschitz condition $|f(t, z_1)-f(t, z_2)| \leqq L_z|z_1-z_2|$ for all (t, z_1), $(t, z_2)\in P_1$, with $L_z<1$. Let $h(t)$ be continuous for $|t| \leqq\alpha$, $h(0)=0$, and $|h(t_1)-h(t_2)| \leqq |t_1-t_2|$ for $t_1, t_2\in[-\alpha, \alpha]$.*

The the problem (5)–(6) *has at least one continuous solution on* $[-\alpha, \alpha]$, *and this is the unique continuous solution on this interval if* α *is sufficiently small.*

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