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ON RINGS WITH A HIGHER DERIVATION

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ABSTRACT. Let $R \supset \mathcal{O}$ be two rings with the unit 1. Then we set $\mathcal{R}(\mathcal{O}, R) = \{x \in R; x^r \in \mathcal{O}$ for some integer $r \geqq 1\}$. At first, it is shown that, under some assumptions, $d\mathcal{O} \subset \mathcal{O}$ implies $d\mathcal{R}(\mathcal{O}, R) \subset \mathcal{R}(\mathcal{O}, R)$. Next, with the Lying-over Theorem on d-differential ideals, we show: Let (R, M) and $(\mathcal{O}, m)$ be two quasi-local rings and let d be a higher derivation of rank ∞ of the total quotient ring of R such that $d\mathcal{O} \subset \mathcal{O}$. Suppose that R is integral over $\mathcal{O}$ and $\mathcal{O}$ is dominated by R. Then $d(m) \subset m$ implies $d(M) \subset M$.

0. Terminology. In this paper, we assume that all rings are commutative and have the unit 1. Let $R \supset \mathcal{O}$ be two rings. Then we set:

$$\mathcal{R}(\mathcal{O}, R) = \{x \in R;\ x^r \in \mathcal{O} \text{ for some integer } r \geqq 1\},$$
$$\mathcal{R}(\mathcal{O}, R)^* = \{x \in R;\ \exists\, y \in \mathcal{O} \text{ such that } yx \in \mathcal{R}(\mathcal{O}, R)\}.$$

Next, let $R^{(s)}$ be the set of s-tuples $(\rho_0, \rho_1, \cdots, \rho_{s-1})$, $\rho_i \in R$, with operations, for $x = (x_0, x_1, \cdots, x_{s-1})$, $y = (y_0, y_1, \cdots, y_{s-1}) \in R^{(s)}$,

$$x + y = (x_0 + y_0, x_1 + y_1, \cdots, x_{s-1} + y_{s-1}),$$
$$xy = (z_0, z_1, \cdots, z_{s-1}), \quad \text{where } z_k = \sum_{i+j=k} x_i y_j.$$

This $R^{(s)}$ is a ring and $R^{(\infty)}$ is isomorphic to a formal power series ring $R[[t]]$ with one indeterminate t over R.

1. On the existence theorems. Let $d = (d_i)_{0 \leq i \leq s-1}$ be a higher derivation of rank s of R in the sense of P. Ribenboim (cf. [1]). Let $\mathfrak{A}$ be an ideal of a ring R. Then we shall call $\mathfrak{A}$ a d-differential ideal if $d\mathfrak{A} \subset \mathfrak{A}$, where $d\mathfrak{A} \subset \mathfrak{A}$ means $d_i\mathfrak{A} \subset \mathfrak{A}$ for all i.

THEOREM 1. *Let d be a higher derivation of rank s (finite or infinite) of R. If $\mathfrak{A}$ is a d-differential ideal of R, then there exists a maximal d-differential ideal $\mathfrak{M}$ of R such that $\mathfrak{M} \supset \mathfrak{A}$.*

PROOF. Let $\mathfrak{F}$ be the set of d-differential ideals containing $\mathfrak{A}$. Then $\mathfrak{F}$ is an inductive set (the order being given by the inclusion relation). Hence Zorn's Lemma implies the existence of $\mathfrak{M}$.

The following proposition is almost the same as Theorem 1 in [3]. But the difference between the two propositions is the definition of higher derivations.

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Proposition 1. *Let $d=(d_i)$ be a higher derivation of a Noetherian ring $\mathfrak{o}$ such that d_0 is an isomorphism. Let $\mathfrak{A}$ be a d-differential ideal of $\mathfrak{o}$, and let $\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_r$ be associated prime ideals of $\mathfrak{A}$. Then $\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_r$ are d-differential ideals and $\mathfrak{A}$ can be written as an irredundant intersection of d-differential primary ideals.*

Proof. We can prove this proposition by almost the same way as Theorem 1 of [3], by considering $\exp(d)$, $\exp(d')$ instead of $\exp(tD)$, $\exp(-tD)$ in [3] respectively, where d' is a higher derivation such that $dd'=1$.

Corollary. *With the same $\mathfrak{o}$, $\mathfrak{A}$ and $d=(d_i)$ as Proposition 1, there exists always a maximal d-differential ideal of $\mathfrak{o}$ containing $\mathfrak{A}$ and this is a prime ideal.*

Theorem 2. *Let $R\supset\mathfrak{o}$ be two rings and let $d=(1, d_1, d_2, \cdots)$ be a higher derivation of the total quotient ring of R such that $d\mathfrak{o}\subset\mathfrak{o}$ and $dR\subset R$. Assume that R is integral over $\mathfrak{o}$. If $\mathfrak{z}$ is a d-differential prime ideal of $\mathfrak{o}$, then there exists a d-differential prime ideal $\mathfrak{z}'$ of R such that $\mathfrak{z}'\cap\mathfrak{o}=\mathfrak{z}$.*

Proof. First, we shall prove that it is sufficient to consider the case when $\mathfrak{o}$ is an d-differential quasi-local ring and the maximal ideal of $\mathfrak{o}$ is d-differential. By the hypothesis, $d(\mathfrak{o}_\mathfrak{z})\subset\mathfrak{o}_\mathfrak{z}$. On the other hand, for $s\in\mathfrak{o}-\mathfrak{z}$, $d_l(1/s)=f(s)/s^{l+1}$, $f(s)\in\mathfrak{o}$. Since $d\mathfrak{z}\subset\mathfrak{z}$, $d(\mathfrak{z}\mathfrak{o}_\mathfrak{z})\subset\mathfrak{z}\mathfrak{o}_\mathfrak{z}$. Hence $\mathfrak{z}\mathfrak{o}_\mathfrak{z}$ is a d-differential ideal and is the maximal ideal of $\mathfrak{o}_\mathfrak{z}$. Let $S=\mathfrak{o}-\mathfrak{z}$. Then R_s is integral over $\mathfrak{o}_s$. Now, suppose that there is a d-differential prime ideal $\mathfrak{M}'$ of R_s such that $\mathfrak{M}'\cap\mathfrak{o}_s=\mathfrak{z}\mathfrak{o}_s$. Let π be the natural mapping $\pi(a)=a\cdot 1$ of $\mathfrak{o}$ into $\mathfrak{o}_s$. Then $\mathfrak{z}'=\pi^{-1}(\mathfrak{M}')$ is a d-differential prime ideal. For let $x\in\mathfrak{z}'$ and $\pi(x)=y\in\mathfrak{M}'$. Then, $\pi(d_i(x))=d_i(x)\cdot 1=d_i((x)\cdot 1)=d_i(\pi(x))$ and $\pi(x)\in\mathfrak{M}'$. Hence $d_i(\pi(x))\in\mathfrak{M}'$ and $d_i(x)\in\mathfrak{z}'$. Trivially, $\mathfrak{z}'\cap\mathfrak{o}\supseteq\mathfrak{z}$. Conversely, let $x\in\mathfrak{z}'\cap\mathfrak{o}$ and $\mathfrak{N}=\mathrm{Ker}(\pi)$. Further let $\pi^*:\mathfrak{o}\rightarrow\mathfrak{o}/\mathfrak{N}$ be the canonical mapping. $\mathfrak{o}/\mathfrak{N}$ is isomorphic to a subring of $\mathfrak{o}_s$. Hence $\pi^*(x)$ is considered as an element of $\mathfrak{o}_s$. Therefore $\pi^*(x)\in\mathfrak{N}'\cap\mathfrak{o}_s$, and $x\in\mathfrak{z}\mathfrak{o}_s$. So, we have $x\in\mathfrak{z}$ and $\mathfrak{z}'\cap\mathfrak{o}=\mathfrak{z}$. Thus we may assume that $\mathfrak{o}$ is a quasi-local ring with a higher derivation d and $\mathfrak{z}$ is the maximal ideal of $\mathfrak{o}$. It is well known that in this case $R\mathfrak{z}\neq R$. Thus, by the Corollary of Proposition 1, there is a maximal d-differential ideal $\mathfrak{M}$ of R such that $\mathfrak{M}\supset R\mathfrak{z}$, and $\mathfrak{M}\cap\mathfrak{o}=\mathfrak{z}$.

2. **On the invariability concerned with a higher derivation.** Let $d=(d_i)$ be a higher derivation of rank s (finite or infinite) of R. Then we introduce the ring homomorphism $\exp(d)$ of $R^{(s+1)}$ into $R^{(s+1)}$ as:

$$\text{for } x = (x_i) \in R^{(s+1)}, \qquad \exp(d)(x) = (z_k) \quad \text{where } z_k = \sum_{i+j=k} d_i(x_j).$$

If d_0 is an isomorphism, then d has inverse higher derivation δ of rank s of R, i.e. $d\delta=1$, $\delta d=1$, where $d\delta=(d_i)(\delta_j)=(\epsilon_k)$, $\epsilon_k=\sum_{i+j=k} d_i\delta_j$. Further,

$$\exp(d\delta) = \exp(d)\exp(\delta) = 1, \qquad \exp(\delta d) = \exp(\delta)\exp(d) = 1.$$

Hence $\exp(d)$ is an isomorphism [**1**].

First, we extend a theorem first proved by A. Seidenberg [**2**, Theorem 1] to the case of a higher derivation in a sense of P. Ribenboim [**1**].

PROPOSITION 2. *Let $R \supset \mathcal{O}$ be two rings and let $\mathcal{O}'$ be the quasi-integral closure (or the complete integral closure) of $\mathcal{O}$ in R. If d is a higher derivation of rank ∞ of R such that $d\mathcal{O} \subset \mathcal{O}$, then $d\mathcal{O}' \subset \mathcal{O}'$.*

PROOF. R is considered as a subring of $R^{(\infty)}$ by the mapping: $x \to (x, 0, 0, \cdots)$. Let α be an element of $\mathcal{O}'$. Then there is an element $\beta \in \mathcal{O}$ such that $\beta\alpha^\rho \in \mathcal{O}$ for all $\rho \geqq 0$. Now, because $\exp(d)(\mathcal{O}) \subset \mathcal{O}^{(\infty)}$,

$$\exp(d)(\beta\alpha^\rho) = \exp(d)(\beta)[\exp(d)(\alpha)]^\rho \in \mathcal{O}^{(\infty)} \quad \text{for all } \rho \geqq 0.$$

Hence $d_0(\beta)d_0(\alpha)^\rho \in \mathcal{O}$ for all $\rho \geqq 0$. On the other hand, $d_0(\beta) \in \mathcal{O}$. Thus, we see that $d_0(\alpha) \in \mathcal{O}'$. Assume that $d_i(\alpha) \in \mathcal{O}'$ for $i \leqq N-1$. Then,

$$d_0(\beta)^N \exp(d)(\beta)[\exp(d)(\alpha) - (d_0(\alpha), \cdots, d_{N-1}(\alpha), 0, 0, \cdots)]^\rho$$
$$= \Big(\underbrace{0, \cdots, 0}_{\rho N-1}, d_0(\beta)^{N+1} d_N(\alpha)^\rho, \cdots\Big) \in \mathcal{O}^{(\infty)}.$$

Therefore, by the induction assumption, $d_0(\beta)^{N+1}d_N(\alpha)^\rho \in \mathcal{O}$ for all $\rho \geqq 0$, and $d_N(\alpha) \in \mathcal{O}'$. This completes the proof.

Next, we shall study the relation of $\mathcal{R}(\mathcal{O}, R)$ and a higher derivation.

THEOREM 3. *Let k be a field of characteristic 0 and let $d=(d_i)$ be a higher derivation of rank ∞ of a domain R ($\supset k$) such that d_0 is an isomorphism. Then $dk \subset k$ implies $d(\mathcal{R}(k, R)) \subset \mathcal{R}(k, R)$.*

PROOF. d can be extended to a higher derivation of K (= the quotient field of R). We shall denote by the same d this extended higher derivation. Let $0 \neq x \in \mathcal{R}(k, R)$. Then there is an integer $r \geqq 1$ such that $x^r \in k$. By the assumption, $d_i(x^r) \in k$ for all i. Hence, $\exp(d)(x^r) = [\exp(d)(x)]^r \in k^{(\infty)}$. Thus $d_0(x)^r \in k$ and $d_0(x) \in \mathcal{R}(k, R)$. Now,

$$\exp(d)(x^r) - (d_0(x), 0, 0, \cdots)^r$$
$$= (0, d_1(x), \cdots)(rd_0(x)^{r-1}, \cdots) \in k^{(\infty)}.$$

Hence $rd_0(x)^{r-1}d_1(x) \in k$. As d_0 is an isomorphism and $d_0(x) \neq 0$, $d_0(x)^{-1} \in K$. Further, $d_0(x)^r \in k$ and $[d_0(x)^r]^{-1} \in k$. Thus $d_0(x)^{-1} = d_0(x)^{r-1}[d_0(x)^r]^{-1} \in R$. Therefore $d_0(x)^{-1} \in \mathfrak{R}(k, R)$. Assume that $d_i(x) \in \mathfrak{R}(k, R)$, $i \leqq N-1$. Then,

$$\exp(d)(x^r) - (d_0(x), d_1(x), \cdots, d_{N-1}(x), 0, 0, \cdots)^r$$
$$= (0, \cdots, 0, rd_0(x)^{r-1}d_N(x), \cdots) \in \mathfrak{R}(k, R)^{(\infty)}.$$

Hence $rd_0(x)^{r-1}d_N(x) \in \mathfrak{R}(k, R)$ and $d_N(x) \in \mathfrak{R}(k, R)$. This completes the proof of our assertion $d(\mathfrak{R}(k, R)) \subset \mathfrak{R}(k, R)$.

COROLLARY. *Under the assumptions of Theorem* 3, $\mathfrak{R}(k, R)$ *is a field.*

THEOREM 4. *Let* $R \supset \mathfrak{O}$ *be two domains and let* $d = (d_i)$ *be a higher derivation of rank* ∞ *of* R. *If* $d\mathfrak{O} \subset \mathfrak{O}$ *and* $d(\mathfrak{R}(\mathfrak{O}, R)) \subset \mathfrak{R}(\mathfrak{O}, R)$, *then* $d(\mathfrak{R}(\mathfrak{O}, R)^*) \subset \mathfrak{R}(\mathfrak{O}, R)^*$.

PROOF. Let $x \in \mathfrak{R}(\mathfrak{O}, R)^*$. Then there is an element $y \in \mathfrak{O}$ such that $yx \in \mathfrak{R}(\mathfrak{O}, R)$. By the hypothesis, $d_0(yx) = d_0(y)d_0(x) \in \mathfrak{R}(\mathfrak{O}, R)$ and $d_0(y) \in \mathfrak{O}$. Hence $d_0(x) \in \mathfrak{R}(\mathfrak{O}, R)^*$. Now, $d_0(y)d_1(yx) = d_0(y)^2d_1(x) + d_0(y)d_0(x)d_1(y)$, $d_0(y)d_1(yx) \in \mathfrak{R}(\mathfrak{O}, R)$ and $d_0(y)d_0(x)d_1(y) \in \mathfrak{R}(\mathfrak{O}, R)$. Therefore $d_0(y)^2d_1(x) \in \mathfrak{R}(\mathfrak{O}, R)$ and $d_1(x) \in \mathfrak{R}(\mathfrak{O}, R)^*$. Assume that $d_i(x) \in \mathfrak{R}(\mathfrak{O}, R)^*$ and $d_i(x)d_0(y)^{r+1} \in \mathfrak{R}(\mathfrak{O}, R)$ for all $i \leqq N-1$. We have

$$d_0(y)^N d_N(yx) = d_0(y)^{N+1}d_N(x) + d_0(y)^N \sum_{i+j=N;\, i\geqq 1} d_i(y)d_j(x).$$

Hence $d_0(y)^{N+1}d_N(x) \in \mathfrak{R}(\mathfrak{O}, R)$, by the induction assumption and the fact that $d_N(x) \in \mathfrak{R}(\mathfrak{O}, R)^*$. Thus we have $d(\mathfrak{R}(\mathfrak{O}, R)^*) \subset \mathfrak{R}(\mathfrak{O}, R)^*$.

COROLLARY 1. *Let* $R \supset \mathfrak{O}$ *be two rings and let* $d = (d_i)$ *be a higher derivation of rank* $s < \infty$ *of* R. *If* $d\mathfrak{O} \subset \mathfrak{O}$ *and* $d(\mathfrak{R}(\mathfrak{O}, R)) \subset \mathfrak{R}(\mathfrak{O}, R)$, *then for any* $x \in \mathfrak{R}(\mathfrak{O}, R)^*$, *there exists a common element* $y \in \mathfrak{O}$ *such that* $yd_i(x) \in \mathfrak{R}(\mathfrak{O}, R)$ *for all* i.

COROLLARY 2. *Under the assumptions of Theorem* 3, $dk \subset k$ *implies* $d(\mathfrak{R}(k, R)) \subset \mathfrak{R}(k, R)$ *and* $d(\mathfrak{R}(k, R)^*) \subset \mathfrak{R}(k, R)^*$.

PROPOSITION 3. *Let* $R \supset \mathfrak{O}$ *be two domains and let* $\mathfrak{O}$ *contain the rational number field. Further, let* $d\mathfrak{O} \subset \mathfrak{O}$, *then, for any invertible element* x *of* $\mathfrak{R}(\mathfrak{O}, R)$, $d_i(x) \in \mathfrak{R}(\mathfrak{O}, R)$ *for all* i.

PROOF. $x \in \mathfrak{R}(\mathfrak{O}, R)$ implies $x^r \in \mathfrak{O}$ for some integer $r \geqq 1$. By the assumption,

$$\exp(d)(x^r) = [\exp(d)(x)]^r = (d_0(x), d_1(x), \cdots)^r \in \mathfrak{O}^{(\infty)}.$$

Hence $d_0(x)^r \in \mathfrak{O}$, and $d_0(x) \in \mathfrak{R}(\mathfrak{O}, R)$. As x is an invertible element, there is a unique element $x^{-1} \in \mathfrak{R}(\mathfrak{O}, R)$ such that $xx^{-1}=1$. From the above discussion, $d_0(x) \in \mathfrak{R}(\mathfrak{O}, R)$. Since d_0 is a homomorphism, $d_0(x)$ is an invertible element of $\mathfrak{R}(\mathfrak{O}, R)$. Now, the tth component of $(d_0(x), d_1(x), \cdots, d_{t-1}(x), \cdots)^r$ is

$$\binom{r}{1} d_0(x)^{r-1} d_t(x) + \Delta(d_0(x), d_1(x), \cdots, d_{t-1}(x))$$

where $\Delta(d_0(x), d_1(x), \cdots, d_{t-1}(x))$ is a polynomial in the $d_0(x)$, $d_1(x), \cdots, d_{t-1}(x)$ with rational coefficients. Assume that, for $i \leqq N-1$, $d_i(x) \in \mathfrak{R}(\mathfrak{O}, R)$. Then, $\binom{r}{1} d_0(x)^{r-1} d_t(x) \in \mathfrak{R}(\mathfrak{O}, R)$. Since $\mathfrak{O}$ contains rational numbers and $d_0(x)^{-1} \in \mathfrak{R}(\mathfrak{O}, R)$, $d_t(x) \in \mathfrak{R}(\mathfrak{O}, R)$. Hence for all $j \geqq 0$, $d_j(x) \in \mathfrak{R}(\mathfrak{O}, R)$. This completes the proof.

LEMMA 1. *Assume that a quasi-local ring* $(\mathfrak{O}, m)$ *is dominated by another quasi-local ring* (R, M). *Then* $\mathfrak{R}(\mathfrak{O}, R)$ *is a quasi-local ring.*

PROOF. It is sufficient to prove that $m = M \cap \mathfrak{R}(\mathfrak{O}, R)$ is the unique maximal ideal of $\mathfrak{R}(\mathfrak{O}, R)$. Assume that $x \in \mathfrak{R}(\mathfrak{O}, R) - m$. Then, for some integer $r \geqq 1$, $x^r \in \mathfrak{O}$ and $x^r \notin m$. Hence $x^{-r} \in \mathfrak{O}$ and $x^{-1} \in R$. Thus $x^{-1} \in \mathfrak{R}(\mathfrak{O}, R)$.

THEOREM 5. *Let* (R, M) *and* $(\mathfrak{O}, m)$ *be two quasi-local rings and let* $d = (1, d_1, d_2, \cdots)$ *be a higher derivation of rank* ∞ *of the total quotient ring of* R *such that* $d\mathfrak{O} \subset \mathfrak{O}$. *Suppose that* R *is integral over* $\mathfrak{O}$ *and* $\mathfrak{O}$ *is dominated by* R. *Then* $dm \subset m$ *implies* $dM \subset M$.

PROOF. By virtue of Theorem 2, there exists a prime ideal M' of R such that $M' \cap \mathfrak{O} = m$ and $dM' \subset M'$. On the other hand, $M \supset M'$. Hence $M \cap \mathfrak{O} \supset M' \cap \mathfrak{O} = m$. By the assumption, $M \cap \mathfrak{O} = m$. Hence $M = M'$. Thus we have $dM \subset M$.

COROLLARY 1. *Let* (R, M) *and* $(\mathfrak{O}, m)$ *be two quasi-local rings and let* $d = (1, d_1, d_2, \cdots)$ *be a higher derivation of rank* ∞ *of the total quotient ring of* R *such that* $d\mathfrak{O} \subset \mathfrak{O}$. *Assume that* $\mathfrak{O}$ *is dominated by* R. *Then* $d(M \cap \mathfrak{R}(\mathfrak{O}, R)) \subset M \cap \mathfrak{R}(\mathfrak{O}, R)$.

PROOF. The first half is the consequence of Lemma 1, and the second half is proved the same way as Theorem 5.

COROLLARY 2. *Under the assumption of Corollary* 1, *let* $\mathfrak{O}$ *contain the rational number field. Then* $d(\mathfrak{R}(\mathfrak{O}, R)) \subset \mathfrak{R}(\mathfrak{O}, R)$.

PROOF. It follows obviously from Proposition 3 and Corollary 1.

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