PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 1, September 1971

GENERALISATION OF THE MUIRHEAD-RADO INEQUALITY

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ABSTRACT. For polynomials $f_\beta(x)$ of n real variables $x = (x_1, x_2, \cdots, x_n)$ of the form

$$f_\beta(x) = \sum_i \sum_j x_{\rho(i,1)}^{\beta_1 e_{j1}} x_{\rho(i,2)}^{\beta_2 e_{j2}} \cdots x_{\rho(i,n)}^{\beta_n e_{jn}},$$

conditions are given which ensure that $f_\alpha(x) \leqq f_\beta(x)$ for all $x \geqq 0$.

Let n be a positive integer and N the set $\{1, 2, \cdots, n\}$. Let G be a group of permutations of N; so if there are g permutations in G we may denote them as $\{(\rho_{i1}, \rho_{i2}, \cdots, \rho_{in}), 1 \leqq i \leqq g\}$. We write $\rho(i, k)$ or ρik for the integer ρ_{ik} of N. Let V be an unordered set of v ordered n-tuples $V = \{(e_{j1}, e_{j2}, \cdots, e_{jn}), 1 \leqq j \leqq v\}$ of real numbers e_{jk}. We say that V is *unaltered* by G if it is unaltered by permuting the elements of every one of its n-tuples by any permutation of G. If $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ is an ordered n-tuple of real numbers β_k we write $H_G(\beta)$ for the convex hull of the set $\{(\beta_{\rho i1}, \beta_{\rho i2}, \cdots, \beta_{\rho in}), 1 \leqq i \leqq g\}$. Finally if $x = (x_1, x_2, \cdots, x_n)$ is an ordered n-tuple of positive real variables $x_k > 0$ we put

$$f_{G,V,\beta}(x) = \sum_{1 \leqq i \leqq g} \sum_{1 \leqq j \leqq v} x_{\rho i1}^{\beta_1 e_{j1}} x_{\rho i2}^{\beta_2 e_{j2}} \cdots x_{\rho in}^{\beta_n e_{jn}}. \tag{1}$$

The object of this note is to prove the

THEOREM. *If V is unaltered by G and if $\alpha \in H_G(\beta)$ then*

$$f_{G,V,\alpha}(x) \leqq f_{G,V,\beta}(x) \qquad \textit{for all } x > 0. \tag{2}$$

PROOF. That $\alpha \in H_G(\beta)$ means [1, §6] that there exist real numbers $t_h \geqq 0$ with

$$\sum_{1 \leqq h \leqq g} t_h = 1 \tag{3}$$

such that

$$\alpha_k = \sum_{1 \leqq h \leqq g} t_h \beta_{\rho hk} \qquad \text{for } 1 \leqq k \leqq n. \tag{4}$$

Received by the editors August 11, 1970.

AMS 1969 *subject classifications*. Primary 2652, 6070, 5230.

Key words and phrases. Convex hull, inequality, Muirhead, permutation group, polynomial, Rado, symmetric function.

Thus

$$f_{G,V,\alpha}(x) = \sum_i \sum_j \prod_h \left(x_{\rho i1}^{\beta_{\rho h1}e_{j1}} \cdots x_{\rho in}^{\beta_{\rho hn}e_{jn}}\right)^{t_h} \tag{5}$$

$$\leqq \sum_i \sum_j \sum_h t_h\left(x_{\rho i1}^{\beta_{\rho h1}e_{j1}} \cdots x_{\rho in}^{\beta_{\rho hn}e_{jn}}\right) \tag{6}$$

$$= \sum_h t_h \sum_i \sum_j x_{\rho(i,\rho^{-1}(h,1))}^{\beta_1 e_{j\rho^{-1}(h,1)}} \cdots x_{\rho(i,\rho^{-1}(h,n))}^{\beta_n e_{j\rho^{-1}(h,n)}} \tag{7}$$

$$= \sum_h t_h \sum_i \sum_j x_{\rho i1}^{\beta_1 e_{j\rho^{-1}(h,1)}} \cdots x_{\rho in}^{\beta_n e_{j\rho^{-1}(h,n)}} \tag{8}$$

$$= \sum_h t_h f_{G,V,\beta}(x) \tag{9}$$

$$= f_{G,V,\beta}(x). \tag{10}$$

To get (5) we substitute (4) in $f_{G,V,\alpha}(x)$, which is similar to (1). Then (6) follows by multiple application of the generalised arithmetic-geometric mean inequality [**2**, Theorem 45]. We get equality in (2) only if we get equality in each of the applications of the inequality. To get (7) from (6) we permute the factors in the bracket by the inverse ρ^{-1} of $(\rho_{h1}, \cdots, \rho_{hn})$. Since G is a group, this inverse lies in G, and applying this inverse to every permutation of G does not change G, hence we get (8). Because V is unaltered by G, the summation over j in (8) is simply summation over all the n-tuples of V, and so by inspection of (1) we get (9). Then (10) follows from (3), and the theorem is proved.

When V is the set of all distinct permutations of n given real numbers, it is clear that V is unaltered by any G. This is true in particular when $1 \leqq m \leqq n$ and V is the set of all $\binom{n}{m}$ permutations of the n-tuple $(1, 1, \cdots, 1, 0, \cdots, 0)$ with m ones. Let $S_m(x_1, \cdots, x_n)$ denote the mth elementary symmetric function on $x_1, \cdots, x_n$, so $S_1 = x_1 + x_2 + \cdots + x_n$ and $S_n = x_1 x_2 \cdots x_n$; this case is the

COROLLARY. *If* $\alpha \in H_G(\beta)$ *then*

$$\sum_i S_m(x_{\rho i1}^{\alpha_1}, \cdots, x_{\rho in}^{\alpha_n}) \leqq \sum_i S_m(x_{\rho i1}^{\beta_1}, \cdots, x_{\rho in}^{\beta_n}) \qquad \textit{for all } x > 0. \tag{11}$$

The case $m = n$ of the corollary was proved by R. Rado [**4**], thus generalising the case $m = n$ and G the group of all permutations which was proved by R. F. Muirhead [**3**] fifty years earlier. Both of them proved for their case that (11) implied $\alpha \in H_G(\beta)$, but this is not true in general. For example, if G is the group of all permutations, and V looks like the unit $n \times n$ matrix, the theorem tells that

$$\alpha \in H_G(\beta) \Rightarrow \sum_{k=1}^{n} y^{\alpha_k} \leqq \sum_{k=1}^{n} y^{\beta_k} \qquad \text{for all } y > 0.$$

However, with $n=4$ we have $(14, 8, 2, 1) \notin H_G(15, 6, 4, 0)$, as can be checked from [4], (3), even though

$$\begin{aligned} &y^{15} + y^6 + y^4 + y^0 - (y^{14} + y^8 + y^2 + y^1) \\ &\quad = \tfrac{1}{4}(y-1)^2\{4y^{13} + 4y^{12} + 3y^{11} + 4y^{10} + 2y^9 + 2y^8 + 4y^7 + y \\ &\qquad + y(y^5-1)^2 + 2(y+1)(y^4-1)^2 + 2(y^3-1)^2\} \geqq 0 \\ &\qquad\qquad \text{for all } y > 0. \end{aligned}$$

Other inequalities of this nature are given in [5].

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