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COUNTABLE PARACOMPACTNESS IN PRODUCT SPACES

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ABSTRACT. The main purpose of this paper is to show that X^ω is normal if and only if (1) X^n is normal for each n, and (2) X^ω is countably paracompact. Furthermore, X^ω is perfectly normal if and only if X^ω is hereditarily countably paracompact. Also, the compact Hausdorff space X is metrizable if and only if X^3 is hereditarily countably paracompact.

This note was prompted by [4], where Michael offers an example of a space X such that for each n, X^n is paracompact, but X^ω is not normal. Following the current notation, X^n denotes the topological product of n copies of X and X^ω denotes the product of countably many copies of X. For each integer n, $\mathcal{P}_n$ denotes the projection of X^ω onto X^n. Throughout this paper, our spaces are assumed to be Hausdorff. The following lemma is due to Ishikawa [2]:

LEMMA A. *The space X is countably paracompact if and only if for each monotonic decreasing sequence $\{H_n\}$ of closed sets with no common part there is a sequence $\{D_n\}$ of open sets such that $H_n \subset D_n$ for each n and such that $\bigcap_{n=1}^\infty \operatorname{cl} D_n = \varnothing$.*

The next lemma was established in [5]:

LEMMA B. *The space X is normal if for each pair (H, K) of mutually exclusive closed subsets of X there is a sequence $\{D_n\}$ of domains such that $H \subset \bigcap_{n=1}^\infty D_n \subset \bigcap_{n=1}^\infty \operatorname{cl} D_n \subset X - K$.*

THEOREM A. *X^ω is normal if and only if* (1) *X^n is normal for each n and* (2) *X^ω is countably paracompact.*

PROOF. First, suppose that each X^n is normal and X^ω is countably paracompact. Let H and K denote mutually exclusive closed subsets of X^ω. The object is to obtain a sequence of domains, $\{D_n\}$, each element of which contains H such that $\{D_n\}$ satisfies our Lemma B. Note that if for some integer n, $\operatorname{cl} \mathcal{P}_n(H) \cap \operatorname{cl} \mathcal{P}_n(K) = \varnothing$, then there are mutually exclusive open sets U and V in X^n such that U contains $\operatorname{cl} \mathcal{P}_n(H)$ and V contains $\operatorname{cl} \mathcal{P}_n(K)$. For each integer j we may let $D_j = \mathcal{P}_n^{-1}(U)$ and we are through; and so, suppose that, for each n,

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cl $\mathcal{P}_n(H) \cap$ cl $\mathcal{P}_n(K) \neq \emptyset$. For each integer n, let $L_n =$ cl $\mathcal{P}_n(H)$ $\cap$cl $\mathcal{P}_n(K)$. Then $\{\mathcal{P}_n^{-1}(L_n)\}$ is a monotonic decreasing sequence of closed subsets of X^ω with no common part; and so, according to Lemma A, there is a sequence $\{U_n\}$ of open subsets of X^ω such that (1) for each n, $\mathcal{P}_n^{-1}(L_n) \subset U_n$ and (2) $\bigcap_{i=1}^\infty$ cl $U_i = \emptyset$. For each n, let $V_n = \mathcal{P}_n(U_n)$. Then $\bigcap_{i=1}^\infty$ cl $\mathcal{P}_n^{-1}(V_n) = \emptyset$ and for each n, V_n contains L_n. Now, for each integer n, [cl $\mathcal{P}_n(H) - V_n$]$\cap$[cl $\mathcal{P}_n(K) - V_n$]$=\emptyset$; and so, there is an open set W_n in X^n such that cl $\mathcal{P}_n(H) - V_n \subset W_n$ $\subset$cl $W_n \subset$cl $\mathcal{P}_n(K) - V_n$. For each integer n, let $D_n = \mathcal{P}_n^{-1}(W_n \cup V_n)$. Clearly, for each integer n, $H \subset D_n$. It remains to show that if x is a point of K, then x is not in $\bigcap_{i=1}^\infty$ cl D_i. To this end, suppose that $x \in [\bigcap_{i=1}^\infty \text{cl } D_i] \cap K$. There is an integer n so that x is not in cl $\mathcal{P}_n^{-1}(V_n)$. Thus, $\mathcal{P}_n(x)$ is in cl $\mathcal{P}_n(K) - V_n$. It follows that $\mathcal{P}_n(x)$ is not in cl W_n; and so, x is not in cl D_n which is a contradiction from which it follows that $K \cap [\bigcap_{i=1}^\infty \text{cl } D_i] = \emptyset$.

Now, suppose that X^ω is normal but X^ω is not countably paracompact. According to our Lemma A, there is a monotonic decreasing sequence $\{H_n\}$ of closed subsets of X^ω with no common part such that if $\{D_n\}$ is a sequence of open sets such that D_n contains H_n for each n, then $\bigcap$cl $D_n \neq \emptyset$. Let x and y denote distinct parts of X and let x^ω (y^ω) denote that point of X each coordinate of which is x (y, resp.). Let

$$H = \bigcup_{n=1}^\infty [(\text{cl } \mathcal{P}_n(H_n)) \times x^\omega] \quad \text{and} \quad K = \bigcup_{n=1}^\infty [(\text{cl } \mathcal{P}_n(H_n)) \times y^\omega].$$

Then H and K are mutually exclusive closed subsets of X^ω. Since X^ω is normal, there are open sets U and V in X^ω such that cl $U \cap$cl $V = \emptyset$ and such that $H \subset U$ and $K \subset V$. Note that for each n, cl $\mathcal{P}_n(H_n) \subset \mathcal{P}_n U$ and cl $\mathcal{P}_n(H_n) \subset \mathcal{P}_n V$. For each integer n, let $D_n = \mathcal{P}_n^{-1}(\mathcal{P}_n U \cap \mathcal{P}_n V)$. Now, $\bigcap_{n=1}^\infty$ cl $D_n \subset$cl$(U \cap V) = \emptyset$ which is a contradiction from which the theorem follows. The following lemma was proved in [5]:

LEMMA C. *The space X is perfectly normal if and only if for each closed set H there is a sequence $\{D_n\}$ of open sets such that $H = \bigcap_{n=1}^\infty D_n = \bigcap_{n=1}^\infty \text{cl } D_n$.*

Theorems B–D (and the methods of proof) should be compared to the results in [3].

THEOREM B. *If $X \times Y$ is hereditarily countably paracompact, then either Y is perfectly normal or every countable discrete subspace of X is closed in X.*

PROOF. Suppose that $X \times Y$ is hereditarily countably paracompact and $M = \{x_1, x_2, \cdots\}$ is a countable discrete subspace of X with a limit point x_0. Let H denote a closed subset of Y. Let $M_1 = \{(x, y) \subset X \times Y \mid x = x_0, y \in Y - H\}$ and $Z = (M \times Y) \cup M_1$. For each integer n, let $K_n = \{(x, y) \in Z \mid x = x_n, y \in H\}$ and $H_n = \bigcup_{i=n}^{\infty} K_i$. Then $\{H_n\}$ is a monotonic decreasing sequence of closed subsets of Z with no common part. Since Z is countably paracompact, by Lemma A, there is a sequence $\{U_n\}$ of open sets in Z such that $\bigcap_{i=1}^{\infty} \mathrm{cl}\ U_i = \varnothing$ and $H_n \subset U_n$ for each n. For each integer n, let $D_n = \{y \in Y \mid (x_n, y) \in \bigcap_{i=1}^{n} U_i\}$. Clearly, $H \subset D_n$ for each n. Suppose that $y \in (Y - H) \cap \bigcap_{i=1}^{\infty} \mathrm{cl}\ D_i$. Then (x_0, y) is a point of Z that is in $\bigcap_{i=1}^{\infty} \mathrm{cl}\ U_i$ which is a contradiction. The theorem now follows from Lemma C.

THEOREM C. *The compact space X is metrizable if and only if X^3 is hereditarily countably paracompact.*

PROOF. It need only be shown that if X^3 is hereditarily countably paracompact, then X is metrizable. Since X^3 is homeomorphic to $X \times X^2$, it follows from Theorem B that X^2 is perfectly normal. Thus, X is metrizable since it is compact and has a G_δ-diagonal.

THEOREM D. *X^ω is perfectly normal if and only if X^ω is hereditarily countably paracompact.*

PROOF. Since any perfectly normal space is hereditarily countably paracompact [**3**], we need only show that if X^ω is hereditarily countably paracompact, then X^ω is perfectly normal. To this end, let x and y denote distinct points of X. For each integer n, let $Z_n = x^n \times y^\omega$; i.e., each of the first n coordinates is x and each remaining coordinate is y. Then the set $\{Z_1, Z_2, \cdots\}$ is a discrete subspace of X^ω that is not closed in X^ω. Now, for each n, X^ω is homeomorphic to $X^\omega \times X^n$; and so, by Theorem B, for each n, X^n is perfectly normal. Thus, by [**3**, Theorem 2], X^ω is perfectly normal.

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