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REALCOMPACTNESS AND PARTITIONS OF UNITY[1]

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ABSTRACT. A characterization of realcompactness in terms of locally finite open coverings and associated partitions of unity is given. Another proof of two well-known theorems of Katětov and Shirota is obtained.

Introduction. Partitions of unity on a topological space X have been extensively studied in the case that X is paracompact, since E. Michael's result [**Mi**, Proposition 2] showed that a paracompact space is characterized by the existence of a wide enough class of partitions of unity. Partitions of unity on a general topological space X have been used, without explicit mention, to prove a key lemma in the book [**GJ**, Lemma 13.7]. Our purpose in this work is to study some relations between ideals of $C(X)$ and partitions of unity on X and give a characterization of realcompactness analogous to that given by Michael for paracompactness. As a by-product, we obtain a proof of Katětov's result: A paracompact T_1 space in which every closed discrete subset has nonmeasurable cardinal, is realcompact. Also, we get another proof of Shirota's theorem: Under the same non-measurability assumption, a $T_{3\frac{1}{2}}$ space is realcompact if and only if it admits a complete uniformity.

1. If X is a topological space, $C(X)$ denotes the ring of real valued continuous functions on X. Terminology and notations concerning $C(X)$ are taken from [**GJ**], while paracompactness and complete regularity are intended in the sense of [**Ke**].

A partition of unity of a space X is a set $\Phi \subset C(X)$ such that:

(a) $\phi \geqq 0$ for every $\phi \in \Phi$;

(b) $\sum_{\phi \in \Phi} \phi(x) = 1$ for all $x \in X$.

A partition of unity Φ which satisfies the further assumption:

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(c) $\{\operatorname{coz} \phi = \phi^{\leftarrow}[R\setminus\{0\}] : \phi \in \Phi\}$ is a locally finite cover of X; is said to be locally finite.

If F is a partition of unity on a space X, then there exists a locally finite partition of unity Φ such that every $\phi \in \Phi$ is a multiple (in $C(X)$) of some $u \in F$ (argue as in [**E**, Lemma 2, pp. 208–209]: if $g(x) = \sup_{u\in F} u(x)$, we have $g \in C(X)$, so that, for every $u \in F$, $f_u = (u - \frac{1}{2}g) \vee 0 \in C(X)$; $\{\operatorname{coz} f_u : u \in F\}$ is a locally finite cover of X; and $\boldsymbol{Z}(f_u)$ is a neighborhood of $\boldsymbol{Z}(u)$, so that f_u is a multiple of u [**GJ**, 1D]. It follows that $\{f_u(\sum_{u\in F} f_u)^{-1} : u \in F\}$ is the required partition of unity).

A partition of unity Φ is said to be minimal if Φ is locally finite and $\phi^{\leftarrow}[1]$ is nonempty for every $\phi \in \Phi$—equivalently, if $\{\operatorname{coz} \phi : \phi \in \Phi\}$ is a minimal (i.e. irreducible) cover of X. By Zorn's lemma, every locally finite cover of a space X has a minimal subcover; hence, if Φ is a locally finite partition of unity, there exists $\Phi_0 \subset \Phi$ such that $\{\phi(\sum_{\psi\in\Phi_0} \psi)^{-1} : \phi \in \Phi_0\}$ is a minimal partition of unity. We have proved the following:

LEMMA. *Let X be a topological space, I an ideal of $C(X)$. If I contains a partition of unity, then I contains a minimal partition of unity (in particular, a locally finite partition of unity).*

Every partition of unity is assumed hereafter to be locally finite.

To give an upper estimate of the cardinality of a minimal partition of unity Φ, it suffices to observe that $\mathfrak{U} = \{\phi^{\leftarrow}[(2/3, 1]] : \phi \in \Phi\}$ is a discrete family of (nonempty) open sets, such that $|\mathfrak{U}| = |\Phi|$ (if S is any set, $|S|$ denotes its cardinality).

Hence, if $\mathfrak{m}$ is a cardinal number, and every discrete family of open subsets of X has cardinality $\leqq \mathfrak{m}$, then every minimal partition of unity has cardinality $\leqq \mathfrak{m}$; in particular, this is true if X is a T_1 space in which every closed discrete subset has cardinality at most $\mathfrak{m}$.

2. Any minimal, infinite partition of unity on a space X generates a free ideal of $C(X)$. And if an ideal I of $C(X)$ contains a partition of unity, then I is free. The next theorem gives a necessary and sufficient condition for a free ideal of $C(X)$ to contain a partition of unity. First, let us say that a cover $\mathcal{S}$ of a topological space X is uniform if there exists a continuous pseudometric d on X and a positive number ϵ such that the (open) d-spheres of radius ϵ are a refinement of $\mathcal{S}$.

THEOREM. *Let X be a topological space and I a free ideal of $C(X)$. Consider the following propositions:*

(a) *I contains a partition of unity.*

(b) *The cover $\mathcal{C}(I) = \{\operatorname{coz} u : u \in I\}$ is uniform.*

(c) *For some continuous pseudometric d on X the z-filter $Z[I]$ is not d-Cauchy.*

For every free ideal I of $C(X)$, (a) *and* (b) *are equivalent and imply* (c). *If I is maximal,* (c) *implies* (b), *hence* (a), (b) *and* (c) *are equivalent.*

PROOF. (a) implies (b). Let $\Phi \subset I$ be a partition of unity. For every $x, y \in X$ put $d(x, y) = \sum_{\phi \in \Phi} |\phi(x) - \phi(y)|$, so that d is a continuous pseudometric on X. For every $x \in X$, let U_x be the open d-sphere of center x and radius 1, and let $\phi_1, \cdots, \phi_n$ be the set of all $\phi \in \Phi$ such that $\phi(x) \neq 0$. If $y \in \boldsymbol{Z}(\phi_1) \cap \cdots \cap \boldsymbol{Z}(\phi_n)$, we have $d(x, y) = \sum_{\phi \in \Phi} |\phi(x) - \phi(y)| \geqq \sum_{i=1}^{n} \phi_i(x) = 1$. Thus, $U_x \subset \operatorname{coz}(\phi_1 + \cdots + \phi_n)$, that is, $\{U_x : x \in X\}$ refines $\mathcal{C}(I)$.

(b) implies (a). Let d be a continuous pseudometric on X, ϵ a positive real number such that the open d-spheres of radius ϵ refine $\mathcal{C}(I)$, and let $\mathcal{U}$ be the family of open d-spheres of radius $\epsilon/2$. Since the pseudometric space (X, d) is paracompact (see [**Ke**]), $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$ by d-open sets. And since d-open sets are cozero sets, $\mathcal{V} = \{\operatorname{coz} f : f \in F\}$, with $F \subset C(X)$. For every $f \in F$, there exist $x \in X$ and $u \in I$ such that $\operatorname{coz} f \subset \{y \in X : d(x, y) < \epsilon/2\} \subset \{y \in X : d(x, y) < \epsilon\} \subset \operatorname{coz} u$. This shows that $\boldsymbol{Z}(f)$ is a neighborhood of $\boldsymbol{Z}(u)$; then f is a multiple of u (see again [**GJ**, 1D]), i.e. $f \in I$. It follows that $\{f^2(\sum_{g \in F} g^2)^{-1} : f \in F\}$ is a partition of unity contained in I.

It is clear that (b) implies (c). To conclude, assume that I is maximal, that d is a continuous pseudometric on X and that ϵ is a positive real number such that every $Z \in \boldsymbol{Z}[I]$ has d-diameter greater than ϵ. For every $x \in X$ put $Z_x = \{y \in X : d(x, y) \leqq \epsilon/2\}$; Z_x is a zero set, and its d-diameter is at most ϵ, hence $Z_x \notin \boldsymbol{Z}[I]$. By maximality, there exists $Z \in \boldsymbol{Z}[I]$ such that $Z_x \cap Z = \varnothing$, i.e. $Z_x \subset X \setminus Z \in \mathcal{C}(I)$. Then the family of open d-spheres of radius $\epsilon/2$ refines $\mathcal{C}(I)$.

3. The following theorem is essentially a formulation, in $C(X)$ terminology, of the Proposition 2 of [**Mi**].

THEOREM. *Let X be a completely regular space. The following are equivalent:*

(a) *X is paracompact.*

(b) *Every free ideal of $C(X)$ contains a partition of unity.*

(c) *Every free z-ideal of $C(X)$ contains a partition of unity.*

PROOF. (a) implies (b). Apply Theorem 2; in a paracompact space, every open cover is uniform (see [**Ke**]). Alternatively, use [**Mi**, Proposition 2].

(b) implies (c). Trivial.

(c) implies (a). We show that every open cover $\mathcal{S}$ of X has an open locally finite refinement. We may assume that $\mathcal{S}$ has no finite subcover, and, by complete regularity, we may also suppose $\mathcal{S} = \{\operatorname{coz} f : f \in F\}$, $F \subset C(X)$. Then the family $\{\boldsymbol{Z}(f) : f \in F\}$ generates a free z-filter $\mathfrak{F}$. Let Φ be a (locally finite) partition of unity contained in the z-ideal $\boldsymbol{Z}^{\leftarrow}[\mathfrak{F}]$. For every $\phi \in \Phi$ choose $f_{\phi,1}, \cdots, f_{\phi,n_\phi} \in F$ such that $\boldsymbol{Z}(\phi) \supset \boldsymbol{Z}(f_{\phi,1}) \cap \cdots \cap \boldsymbol{Z}(f_{\phi,n_\phi})$. Hence $\{\operatorname{coz}(\phi f_{\phi,i}) : \phi \in \Phi, i = 1, \cdots, n_\phi\}$ is an open locally finite refinement of $\mathcal{S}$.

4. Theorem 2 shows that a free maximal ideal of $C(X)$ contains a partition of unity if and only if its z-filter contains no small sets. Here we establish an algebraic characterization of maximal ideals which contain a partition of unity.

THEOREM. *Let X be a topological space, M a maximal ideal of $C(X)$, O_M the intersection of all prime ideals contained in M. The following are equivalent.*

(a) *M is hyperreal.*

(b) *O_M contains a countable partition of unity.*

(c) *M contains a partition of unity of nonmeasurable cardinal.*

PROOF. (a) implies (b). The construction used in the proof is taken from [**GJ**, 13.7]. Let $s \in C(X)$ be such that $M(s)$ is infinitely large. For each $n \in Z$ (Z is the set of integers) define $\psi_n \in C(R)$ as follows: The support of ψ_n is $[n-1, n+1]$; $\psi_n(n) = 1$; ψ_n is linear in $[n-1, n]$ and in $[n, n+1]$. Put $\phi_n = \psi_n \circ s$. Then $\{\phi_n : n \in Z\}$ is a partition of unity; and $\boldsymbol{Z}(\phi_n) \supseteq \{x \in X : s(x) \geqq n+1\} = Z_n$. By [**GJ**, 13.7], $Z_n \in \boldsymbol{Z}[O_M]$, that is, $\phi_n \in O_M$ for every $n \in \boldsymbol{Z}$.

(b) implies (c). Obvious.

(c) implies (a). Let Φ be a partition of unity contained in a maximal ideal M of $C(X)$. Suppose that M is real, and consider Φ as a discrete topological space. For every $f \in C(\Phi)$, put $L(f) = M(\sum_{\phi \in \Phi} f(\phi)\phi)$. It is easily seen that L is a positive linear functional on $C(\Phi)$. Furthermore, $L(1) = 1$ (so that $L \neq 0$) and $L(\chi_\phi) = 0$ for every $\phi \in \Phi$ (χ_ϕ is the characteristic function of $\{\phi\}$ in Φ). It follows from [**M**] that $|\Phi|$ is measurable.

5. Theorem 4 gives a characterization of realcompactness which is suggestive of paracompactness.

We define a maximal open cover of a topological space X as an open cover which is maximal in respect to the property of being finitely inadequate (i.e., of having no finite subcover). By Zorn's lemma, every finitely inadequate open cover is contained in a maximal open cover. A cover by cozero sets is said to be basic.

THEOREM. *Let X be a completely regular T_1-space. The following are equivalent.*

(a) *X is realcompact.*

(b) *Every maximal open cover has a locally finite, countable basic refinement.*

(c) *Every maximal open cover has a locally finite basic refinement of nonmeasurable cardinal.*

(d) *Every free maximal ideal of $C(X)$ contains a partition of unity of nonmeasurable cardinal.*

PROOF. (a) implies (b). Let $\mathcal{S}$ be a maximal open cover of X. By complete regularity of X, and maximality of $\mathcal{S}$, the set $\mathfrak{F}=\{Z\in \boldsymbol{Z}(X): X\backslash Z\in\mathcal{S}\}$ is a prime, free z-filter. By the first part of Theorem 4, the prime ideal $\boldsymbol{Z}^{\leftarrow}[\mathfrak{F}]$ contains a countable partition of unity Φ. Thus, $\{\operatorname{coz}\phi:\phi\in\Phi\}$ is the required refinement of $\mathcal{S}$.

(b) implies (c). Trivial.

(c) implies (d). If M is a free ideal of $C(X)$, the open cover $\mathcal{C}(M)=\{\operatorname{coz} u: u\in M\}$ is contained in some maximal open cover $\mathcal{S}$. If $\{\operatorname{coz} f: f\in F\}$, $F\subset C(X)$, $|F|$ nonmeasurable, is a locally finite basic refinement of $\mathcal{S}$, then $\Phi=\{f^2(\sum_{g\in F} g^2)^{-1}: f\in F\}$ is a partition of unity of nonmeasurable cardinal. The maximality of M implies $F\subset M$, hence $\Phi\subset M$.

(d) implies (a). Theorem 4.

6. In this paragraph we present some corollaries of the previous results. By Theorem 5, Theorem 3 and Lemma 1 we obtain Katětov's result [**K**, Theorem 3]:

Let X be a paracompact T_1 space. X is realcompact if and only if every discrete family of open subsets of X has nonmeasurable cardinal.

By Theorem 4, Theorem 2 and Lemma 1 we obtain Shirota's theorem ([**S**]; [**GJ**, 15.21]), which we state in the following form.

Let X be a topological space in which every discrete family of open sets has nonmeasurable cardinal, and let d be a continuous pseudometric on X. Then every real z-ultrafilter is d-Cauchy.

REMARK. Katětov's theorem is a particular case of Shirota's theorem (see [**S**, Remark added in proof]). And in fact, Theorem 3 and Theorem 2 imply that every paracompact space is complete in its largest uniformity.

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