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A CLASS OF ARCWISE CONNECTED CONTINUA

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ABSTRACT. It is known that every bounded semi-aposyndetic plane continuum which does not separate the plane is arcwise connected. To show that this theorem remains true if the phrase "does not separate the plane" is replaced by "does not have infinitely many complementary domains" is the primary purpose of this paper.

Let M be a continuum (a closed connected point set) and let x and y be distinct points of M. If M contains a continuum H and an open set G such that $x \in G \subset H \subset M - \{y\}$, then M is said to be *aposyndetic* at x with respect to y. If M is aposyndetic at x with respect to each point of $M - \{x\}$, then M is said to be *aposyndetic* at x. F. Burton Jones has shown that if M is a bounded plane continuum which is aposyndetic (that is, aposyndetic at each of its points) and does not have infinitely many complementary domains, then M is locally connected and therefore arcwise connected [**4**, Theorem 10]. The notion of aposyndesis can be generalized as follows.

DEFINITION. A continuum M is said to be *semi-aposyndetic* if for each pair of distinct points x and y of M, M is aposyndetic either at x with respect to y or at y with respect to x.

In a recent paper [**3**], the author established the arcwise connectedness property for bounded semi-aposyndetic nonseparating plane continua. For a sketch of the proof of this theorem see [**2**]. There are nonlocally connected continua, hence continua which do not have the conditions specified in Jones's theorem, which have these properties. However, the arcwise connectedness implication of Jones's theorem is not generalized by this result. The purpose of this paper is to extend the author's result to a class of continua which includes those studied by Jones. Here it is proved that if M is a bounded semi-aposyndetic plane continuum which does not have infinitely many complementary domains, then M is arcwise connected.

Throughout this paper S is the set of points of a simple closed surface (that is, a 2-sphere).

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THEOREM 1. *Suppose M is a continuum in S and $S-M$ does not have infinitely many components. Let D be the interior of a disk in S. Suppose F is a component of $M-D$ and x and y are points of F such that F is not aposyndetic at x with respect to y. Then M is not aposyndetic at x with respect to y.*

PROOF. Assume M is aposyndetic at x with respect to y. There exists a continuum H and a region V containing y such that x is in the interior of H relative to M and H is in $M-\text{Cl } V$ (Cl V is the closure of V). Let U_1 and V_1 be circular regions in S having radii less than 1 centered on x and y respectively such that $U_1 \cap M \subset H$, $V_1 \subset V$, and $\text{Cl } U_1 \cap \text{Cl } V_1 = \varnothing$. The component of $F-V_1$ which contains x is not open (relative to F) at x. Hence Bd U_1 (the boundary of U_1) contains an arc-segment I_1 whose endpoints a_1 and b_1 lie in different components of $F-V_1$ such that $F \cap I_1 = \varnothing$. There exists a simple closed curve J_1 which separates a_1 from b_1 and contains no point of $M-(D\cup V_1)$. In $J_1 \cup I_1$ there exists a simple closed curve K_1 which separates a_1 from b_1 and contains no point of $F-V_1$ such that $K_1 \cap I_1$ is connected. There is an arc-segment T_1 in K_1 which crosses I_1, lies in $S-V_1$, and has its endpoints c_1 and d_1 in $\text{Bd } V_1 \cap ((S-M)\cup D)$. Let U_2 and V_2 be circular regions having radii less than $\frac{1}{2}$ centered on x and y respectively such that $\text{Cl } U_2 \subset U_1 - T_1$ and $\text{Cl } V_2 \subset V_1$. The component of $F-V_2$ which contains x is not open (relative to F) at x. Hence there exists an arc-segment I_2 in Bd U_2 whose endpoints a_2 and b_2 lie in different components of $F-V_2$ such that $F \cap I_2 = \varnothing$. There exists a simple closed curve K_2 which separates a_2 from b_2 and contains no point of $F-V_2$ such that $K_2 \cap I_2$ is connected and $(K_2-I_2)\cap(M-(V_2\cup D)) = \varnothing$. In K_2-V_2 there is an arc-segment T_2 which crosses I_2 and has its endpoints c_2 and d_2 in $\text{Bd } V_2 \cap ((S-M)\cup D)$. Continue this process. For each $n=3, 4, 5, \cdots$, there exist circular regions U_n, V_n, arc-segments I_n, T_n, and a simple closed curve K_n such that (1) U_n and V_n are centered on x and y respectively and each have radius less than $1/n$, (2) $\text{Cl } U_n \subset U_{n-1} - \bigcup_{i=1}^{n-1} T_i$ and $\text{Cl } V_n \subset V_{n-1}$, (3) I_n is in $\text{Bd } U_n - F$ and has endpoints a_n and b_n in F, (4) K_n separates a_n from b_n in S, (5) $K_n \subset (S-F)\cup V_n$ and $K_n \cap I_n$ is connected, (6) $T_n \subset K_n - V_n$ and $T_n \cap I_n \neq \varnothing$, and (7) the endpoints c_n and d_n of T_n are in $\text{Bd } V_n \cap ((S-M)\cup D)$.

Since M has only a finite number of complementary domains, it may be assumed without loss of generality that there exist complementary domains C and G of M such that $\{c_1, c_2, c_3, \cdots\}$ is in $C\cup D$ and $\{d_1, d_2, d_3, \cdots\}$ is in $G\cup D$ (here it may be necessary to delete certain elements of the original sequences and rename others). Suppose $\{c_1, c_2, c_3, \cdots\}\cap D$ is infinite. Assume without loss of generality that $\{c_1, c_2, c_3, \cdots\}$ is contained in D. Since $c_1, c_2, c_3, \cdots$ converges to y, y belongs to the boundary of the disk Cl D. There exists a region E in V_1 containing y such that $E\cap D$ is the interior of a disk in S [**6**, Theorem 6, p. 163]. Note that $E\cap D$ contains all but finitely many points of $c_1, c_2, c_3, \cdots$. Suppose, without loss of generality, that $\{c_1, c_2, c_3, \cdots\}$ is a subset of $E\cap D$. The set $D \cap\{d_1, d_2, d_3, \cdots\}$ is finite; for otherwise, there would exist an integer j such that $\{c_j, d_j\}\subset E\cap D$ and $T_j\cup(E\cap D)$ would separate a_j from b_j which contradicts the assumption that F is a component of $M-D$ [**6**, Theorem 32, p. 181]. Furthermore, if i and j are positive integers ($i<j$), then $F\cup I_j$ separates d_i from d_j in S; for if this were not the case, $S-(F\cup I_j)$ would contain an arc A which goes from d_i to d_j and $(E\cap D)\cup A\cup T_i\cup T_j$ would separate a_j from b_j, again contradicting the assumption that F is a component of $M-D$. Hence there exist integers i and j ($i<j$) such that $\{d_i, d_j\}\subset S-M$ and $F\cup I_j$ separates d_i from d_j in S. Since d_i and d_j are in the same complementary domain of M, there exists an arc B in $S-M$ which goes from d_i to d_j. Let Z denote the complementary domain of $F\cup I_j$ which contains d_j. Let k denote the first point of $B\cap\text{Bd } V_i\cap Z$ and let h be the last point of $B\cap\text{Bd } V_i$ which precedes k with respect to the order on B. Let L be the subarc of B which has endpoints h and k. Note that h does not belong to Z and $L\cap\text{Cl } V_i=\{h, k\}$. $(F\cup I_j)-V_i$ separates h from k in $S-V_i$. There exists a continuum N in $(F\cup I_j)-V_i$ which separates h from k in $S-V_i$ [**6**, Theorem 27, p. 177]. Let B_1 and B_2 be mutually exclusive arc-segments in Bd V_i which have endpoints h and k. For $n=1$ and 2, there exists a point e_n in $B_n\cap N$. The points e_1 and e_2 are contained in distinct components of $N-I_j$ [**6**, Theorem 28, p. 156]. It follows that for $n=1$ and 2, $\{a_j, b_j\}$ meets the e_n-component of $N-I_j$ at exactly one point. The θ-curve $L\cup\text{Bd } V_i$ separates a_j from b_j in S [**6**, Theorem 28, p. 156]. But since H is a continuum in $S-(L\cup\text{Bd } V_i)$ containing $\{a_j, b_j\}$, this is a contradiction. Hence $\{c_1, c_2, c_3, \cdots\}\cap D$ is finite.

It follows from the same argument that $\{d_1, d_2, d_3, \cdots\}\cap D$ is finite (the roles of the sequences $c_1, c_2, c_3, \cdots$ and $d_1, d_2, d_3, \cdots$ are interchanged). Hence there exist integers i and j ($i<j$) such that $\{c_i, d_i, c_j, d_j\}$ is contained in $S-M$. By the argument in the last part of the preceding paragraph, $F\cup I_j$ does not separate either c_i from c_j or d_i from d_j in S. There exist arcs X and Y in $S-(F\cup I_j)$ such that X has endpoints c_i and c_j and Y has endpoints d_i and d_j. It follows that $X\cup Y\cup T_i\cup T_j$ contains a simple closed curve J which

separates a_j from b_j [**6**, Theorem 32, p. 181]. Since J does not meet F, this is a contradiction. Hence M is not aposyndetic at x with respect to y.

DEFINITION. A point y of a continuum M *cuts* x from z in M (*cuts* M between x and z) if x, y, and z are distinct points of M and y belongs to each subcontinuum of M which contains $\{x, z\}$.

THEOREM 2. *If M is a bounded semi-aposyndetic plane continuum which does not have infinitely many complementary domains, then M is arcwise connected.*

PROOF. Let P be the set consisting of all natural numbers n such that if M is a semi-aposyndetic continuum in S and $S-M$ has exactly n components, then M is arcwise connected. The natural number 1 belongs to P [**3**]. Assume $1, 2, \cdots, m$ belong to P. Let M be a semi-aposyndetic continuum in S such that $S-M$ has exactly $m+1$ components. Let p and q be distinct points of M. Let G denote a component of $S-M$ and define C to be $S-(M\cup G)$. Suppose $\mathrm{Cl}\, G\cap \mathrm{Cl}\, C$ is totally disconnected. There exists a simple closed curve J in M which separates G from a component of C [**7**, Theorem 3.1, p. 108]. Let K and L be the complementary domains of J. It follows from Theorem 1 that $(\mathrm{Cl}\, K\cap M)\cup L$ and $(\mathrm{Cl}\, L\cap M)\cup K$ are semi-aposyndetic continua in S each having no more than m complementary domains. By the induction assumption, there exist arcs I and B in $(\mathrm{Cl}\, K\cap M)\cup L$ and $(\mathrm{Cl}\, L\cap M)\cup K$ which go from p to q. In $(J\cup I\cup B)\cap M$ there exists an arc with endpoints p and q.

Suppose $\mathrm{Cl}\, G\cap \mathrm{Cl}\, C$ has a nondegenerate component Q. If $Q-\{p, q\}$ contains a point x which does not cut M between p and q, then in S there is a circular region D containing x such that $\{p, q\}$ is a subset of a component F of $M-D$. F is a semi-aposyndetic continuum (Theorem 1) in S which has no more than m complementary domains. Hence F contains an arc with endpoints p and q. Suppose each point of $Q-\{p, q\}$ cuts M between p and q. Let Z be the set of all points which cut p from q in M. There exists an arc A (not necessarily in S) from p to q containing Z such that if x is a point of Z, then x does not cut M either between two points of (p-component of $A-\{x\})\cap(Z\cup\{p\})$ or between two points of (q-component of $A-\{x\})\cap(Z\cup\{q\})$ [**3** (in the theorem's proof)]. Let T be a subarc of $(A-\{p, q\})\cap Q$. Let r and t denote the endpoints of T and assume without loss of generality that t follows r with respect to the order on A. Let x be a point of $T-\{r,t\}$. There exists a circular region U in S containing x such that H (the r-component of $M-U$) contains p and Y (the t-component of $M-U$) contains q. H and Y

are semi-aposyndetic continua (Theorem 1) in S and the sets $S-H$ and $S-Y$ each have no more than m components. It follows that $H\cup Y\cup T$ contains an arc which has endpoints p and q. Evidently M is arcwise connected. Hence $m+1$ belongs to P. By the Second Principle of Induction, all natural numbers belong to P.

REMARK. Let M be a bounded plane continuum which does not separate the plane. In [**5**], it is proved that M has Jones's cyclic property (that is, if p and q are distinct points of M and no point cuts p from q in M, then there exists a simple closed curve lying in M which contains p and q). For a related theorem see [**1**, Theorem 2]. Also in [**5**], there is an example of a bounded plane continuum which has two complementary domains and fails to have this cyclic property. However, this continuum is not semi-aposyndetic. Using Theorem 1 and Theorem 2, one can easily establish Jones's cyclic property for semi-aposyndetic bounded plane continua which do not have infinitely many complementary domains.

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