PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 1, September 1971

ENDOMORPHISMS OF FINITELY PRESENTED MODULES

GABRIEL SABBAGH

ABSTRACT. It is proved that every surjective or injective endomorphism of a finitely presented left module over a right perfect ring is an isomorphism.

We will adopt the following conventions: Rings and modules are unitary. Module means left module, ideal means left ideal.

Let R be a ring and M be a *finitely generated* R-module. It is well known that if R is commutative and if M is free, then any two bases of M have the same number of elements. More generally one has ([9], [10]):

(1) If R is commutative, then every surjective endomorphism of M is an isomorphism.

On the other hand, the following has recently been proved [11]:

(2) If R is commutative and of Krull dimension zero, then every injective endomorphism of M is an isomorphism.

In [11] it is suggested that (2) should be valid for rings which are close to being Artinian, as perfect rings. Our main result is:

THEOREM. *If R is right perfect and if M is finitely presented, then every injective or surjective endomorphism of M is an isomorphism.*

We first need some facts on perfect modules. We recall that a module is said to be *perfect* if it satisfies the descending chain condition on cyclic submodules. The ring R is said to be *right perfect* if the (left) R-module R is perfect ([2], [3]).

The first of the following statements is Theorem 2 of [3]. The others may easily be derived from it or are immediate.

(3) A perfect module satisfies the descending chain condition on finitely generated submodules.

(4) The quotient of a perfect module by a finitely generated submodule is a perfect module.

(5) Every submodule of a perfect module is a perfect module.

(6) The direct sum of an arbitrary family of perfect modules is a perfect module.

Received by the editors October 19, 1970.

AMS 1969 *subject classifications*. Primary 1325; Secondary 1340.

Key words and phrases. Injective endomorphism, surjective endomorphism, isomorphism, finitely presented module, perfect ring.

We state as a lemma the following simple consequence of (4) and (6):

LEMMA 1. *If R is right perfect, then every finitely presented R-module is perfect.*

PROPOSITION. *Every injective endomorphism of a finitely generated perfect module is an isomorphism.*

PROOF. Let X be a finitely generated perfect module and let h be an injective endomorphism of X. The sequence $\langle h^n(X),\ n \geqq 0\rangle$ constitutes a descending chain of finitely generated submodules of X. By (3) there exists an integer n such that $h^n(X) = h^{n+1}(X)$. Since h is injective, one obtains $X = h(X)$.

The next lemma, the first part of which generalizes results of [**6**] and [**7**], is anticipated in an interesting although erroneous footnote of [**5**] and is essentially proved, in rather a different way, in an unpublished dissertation of G. B. Klatt, is of some independent interest.

LEMMA 2. *If R is semiperfect, then*:

(7) *Every projective R-module may be written in a unique way as a direct sum of indecomposable cyclic* (*projective*) *modules.*

(8) *Every surjective endomorphism of a finitely generated projective R-module is an isomorphism.*

PROOF. (7): It is well known (and proved in [**8**]) that if R is semiperfect then the endomorphism ring of every projective indecomposable R-module is local. This fact yields the uniqueness part of (7), thanks to Azumaya's version of the Krull-Schmidt theorem [**1**], and allows us by [**12**] to reduce the remaining part of (7) to the following: If R is semiperfect, then every *free* R-module F is the direct sum of projective indecomposable cyclic modules. It is clearly enough to prove this in the case where F is the R-module R itself. This last step is performed in [**8**].

(8): It is easy to see that the following assertion, which is an immediate consequence of (7), is equivalent to (8): For every finitely generated projective R-module P and every R-module Q the R-modules P and $P \oplus Q$ are isomorphic only if Q is the 0 module.[1]

We now proceed to the *proof* of the theorem. We assume that R is a right perfect ring and that M is a finitely presented R-module. Let h be an endomorphism of M. It follows from Lemma 1 and from the

[1] The referee observed that if that assertion holds for the quotient of an arbitrary ring S by its Jacobson radical then it also holds for S, which gives another proof of (8).

proposition that if h is injective then h is an isomorphism. What remains to be shown is that, assuming that h is surjective, h is an isomorphism.

Let $p: P \to M$ be a projective cover of M [**2**]. By definition p is surjective and, since M is the quotient of a finitely generated projective module, it is easy to see that P is finitely generated. Since P is projective, there exists an endomorphism g of P such that $p \circ g = h \circ p$. Since (P, p) is a projective cover of M and since $p \circ g$ is surjective, it follows that g is surjective. By applying the second part of Lemma 2 we deduce that g is an isomorphism. Let K now be the kernel of p. It follows from [**4**, Lemme 9, p. 37] that K is finitely generated. It is clear that g maps K into K. The restriction g' of g to K is then an injective endomorphism of K. By (5) and Lemma 1 (applied to P which is finitely presented) K is a perfect module. The proposition then implies that g' is an isomorphism of K. It follows that h is injective. The proof is complete.

We conclude the paper with a proposition inspired by (2).

PROPOSITION. *If R is commutative, then every injective pure endomorphism of M is an isomorphism.*

PROOF. Let h be an injective pure endomorphism of M. Our notion of purity is the one introduced by Cohn (see e.g. [**4**, Ex. 24, p. 66]) and allows us to say that the exact sequence $0 \to M \xrightarrow{h} M$ remains exact upon tensoring by any cyclic R-module. For showing that h is surjective, it suffices by [**4**, Proposition 11, p. 113] to show that for every maximal ideal $\mathfrak{M}$ of R the homomorphism $h_{\mathfrak{M}}: M/\mathfrak{M}M \to M/\mathfrak{M}M$ induced by h is surjective. The modules $R/\mathfrak{M} \otimes M$ and $M/\mathfrak{M}M$ are canonically isomorphic and we may identify $h_{\mathfrak{M}}$ with $1 \otimes h: R/\mathfrak{M} \otimes M \to R/\mathfrak{M} \otimes M$. Hence $h_{\mathfrak{M}}$ is injective. Since M is finitely generated, the $R/\mathfrak{M}$-vector space $M/\mathfrak{M}M$ is of finite dimension. It follows that $h_{\mathfrak{M}}$ is surjective and the proof is complete.

Two remarks are in order:

(a) Let N denote the two-sided ideal generated by the nilpotent elements of R. If R is commutative, R is of Krull dimension zero if and only if the ring R/N is a von Neumann regular ring [**4**, Ex. 16, p. 173]. On the other hand, if S is a von Neumann regular ring, every S-module is a pure submodule of every S-module containing it. The connection between (2) and the previous proposition is now clear.

(b) The assumption of the commutativity of R cannot be dispensed with in the previous proposition. Indeed, let V denote a vector space of infinite dimension over an arbitrary field K. Let us take for R the ring of all K-endomorphisms of V. It is easy to see (and well known) that there exists an isomorphism f of the R-module $R\oplus R$ onto the R-module R. Let j denote the embedding: $x\rightarrow(x, 0)$ of R into $R\oplus R$. It is clear that the injective endomorphism $f\circ j$ of R is pure and is not an isomorphism. Note that in that example R is a von Neumann regular ring.

References

1. G. Azumaya, *Corrections and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem.* Nagoya Math. J. **1** (1950), 117–124. MR **12,** 314.

2. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings,* Trans. Amer. Math. Soc. **95** (1960), 466–488. MR **28** #1212.

3. J. E. Björk, *Rings satisfying a minimum condition on principal ideals,* J. Reine Angew. Math. **236** (1969), 112–119. MR **40** #1419.

4. N. Bourbaki, *Algèbre commutative.* Chaps. 1, 2, Actualités Sci. Indust., no. 1290, Hermann, Paris, 1961. MR **36** #146.

5. P. M. Cohn, *Hereditary local rings,* Nagoya Math. J. **27** (1966), 223–230. MR **33** #5663.

6. S. Eilenberg, *Homological dimension and syzygies,* Ann. of Math. (2) **64** (1956), 328–336. MR **18,** 558.

7. I. Kaplansky, *Projective modules,* Ann. of Math (2) **68** (1958), 372–377. MR **20** #6453.

8. J. Lambek, *Lectures on rings and modules,* Blaisdell, Toronto, 1966. MR **34** #5857.

9. J. R. Strooker, *Lifting projectives,* Nagoya Math. J. **27** (1966), 747–751. MR **33** #5679.

10. W. V. Vasconcelos, *On finitely generated flat modules,* Trans. Amer. Math. Soc. **138** (1969), 505–512. MR **39** #199.

11. ———, *Injective endomorphisms of finitely generated modules,* Proc. Amer. Math. Soc. **25** (1970), 900–901.

12. R. B. Warfield, Jr., *A Krull-Schmidt theorem for infinite sums of modules,* Proc. Amer. Math. Soc. **22** (1969), 460–465. MR **39** #4213.

1 Square François Couperin, 92-Antony, France