PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 1, September 1971

A CONVERGENCE QUESTION IN H^p

STEPHEN SCHEINBERG[1]

ABSTRACT. Let $\phi \in H^p$ (unit disc), $0 < p < \infty$. and let $\phi_r(z) = \phi(rz)$, $r < 1$. If ϕ contains a nontrivial inner factor, it is known that ϕ/ϕ_r is unbounded in H^p-norm. We prove that if ϕ is analytic on the closed disc and has no zeros on the open disc, then $\phi/\phi_r \to 1$ in H^p, as $r \to 1$. The same conclusion follows if $1/\phi \in H^\infty$. We construct an outer function ϕ which is continuous on the closed disc, analytic for $z \neq 1$, and such that ϕ/ϕ_r is unbounded in every H^p.

The following question, originating with G. Lumer, I believe, was put to me by L. Zalcman: if $\phi \in H^2$ and is outer, does it follow that $\phi(z)/\phi(rz) \to 1$ in H^2 as $r \to 1^-$? This note provides the following answers to this question.[2] It will be clear that the same results hold in H^p, for $0 < p < \infty$.

THEOREM. (a) *If it is also true that* $|\phi| \geqq \delta > 0$ *on* $\{|z| < 1\}$, *then the answer is "yes".*

(b) *If* ϕ *is actually analytic on* $\{|z| \leqq 1\}$, *then the answer is "yes."*

(c) *There is a* ϕ *which is continuous on* $\{|z| \leqq 1\}$, *is analytic on* $\{|z| \leqq 1, z \neq 1\}$, *is outer and has*

$$\limsup_{r \to 1} \int \left| \frac{\phi(e^{i\theta})}{\phi(re^{i\theta})} \right|^{\epsilon} d\theta = +\infty,$$

for any $\epsilon > 0$. *Hence, in particular,* $\phi(e^{i\theta})/\phi(re^{i\theta}) \not\to 1$ *in* L^2.

The positive results (a) and (b) and the negative result (c) provide natural limitations on each other and are in a certain sense best possible. The basic notions and facts of H^p-theory can be found in Hoffman's book.[3] The proofs of (a), (b), and (c) involve computations;

Received by the editors May 21, 1970.

AMS 1969 *subject classifications.* Primary 3067, 3085.

Key words and phrases. Inner function, outer function.

[1] Preparation of this paper was supported in part by NSF GP 11911.

[2] As the referee points out, part (c) of the Theorem answers in the negative the following natural question in prediction theory. Consider the classical prediction problem for one-parameter stationary processes and obtain the formal expression for the predictor from the Taylor coefficients (at 0) of the inverse of the generating function. Is this formal expression necessarily Abel-summable?

[3] Kenneth Hoffman, *Banach spaces of analytic functions*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N. J., 1962. MR **24** #A2844.

(a) and (b) are straightforward; (c) involves manipulations of the Poisson kernel, and the example is motivated by the inner function $\exp[(z+1)/(z-1)]$, for which the convergence in question is well known to fail. This last fact is the reason for the hypothesis that ϕ be outer.

REMARK.

$$0 \leqq \left|1 - \frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 = 1 - 2\,\mathrm{Re}\,\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})} + \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 .$$

$\mathrm{Re}(\phi(e^{i\theta})/\phi(re^{i\theta}))$ are the boundary values of a harmonic function which is 1 at the origin. Therefore,

$$0 \leqq \int \left|1 - \frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \frac{d\theta}{2\pi} = 1 - 2 + \int \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \frac{d\theta}{2\pi} .$$

Thus, the original question is equivalent to

$$\text{“does } \int_{-\pi}^{\pi} \left|\frac{\phi(e^{i\theta})}{\phi(re^{i\theta})}\right|^2 \frac{d\theta}{2\pi} \to 1?\text{”}$$

When “2” is replaced by “p” ,the same equivalence holds. It is a standard fact of Lebesgue theory that if $f_r \to f$, a.e., and $\int |f_r|^p \to \int |f|^p$, then $f_r \to f$ in L^p.

PROOFS. (a) If $|\phi| \geqq \delta > 0$, then $1/\phi \in H^\infty$ and since $\phi(re^{i\theta}) \to \phi(e^{i\theta})$, a.e., we have $\phi(e^{i\theta})/\phi(re^{i\theta}) \to 1$, a.e., and dominated by the L^2-function const $|\phi(e^{i\theta})|$. The same proof works in H^p.

(b) If ϕ is analytic on $|z| \leqq 1$ and is outer, it has the form $\phi(z) = (z - e^{i\theta_1})^{n_1} \cdots (z - e^{i\theta_q})^{n_q}\psi(z)$, where $\theta_1, \cdots$ are distinct in $0 \leqq \theta < 2\pi$ and both ψ and $1/\psi$ are analytic for $|z| \leqq 1$.

It is sufficient to show $|(1 - e^{i\theta})/(1 - re^{i\theta})|^m \to 1$ in $L^2(d\theta)$, for each $m \geqq 1$. For if this is the case, then $|\phi(e^{i\theta})/\phi(re^{i\theta})| =$ product of terms, each of which converges to 1 in L^2 and pointwise, a.e. On any small interval of θ's, at most one term is unbounded as $r \to 1$. Thus, we will be done by the Remark above. I thank D. Neu for the following argument, which considerably simplifies the proof. Since

$$\left|\frac{1 - e^{i\theta}}{1 - re^{i\theta}}\right| \leqq 2 \quad \text{and} \quad \frac{1 - e^{i\theta}}{1 - re^{i\theta}} \to 1, \quad \text{a.e.,}$$

the Lebesgue bounded convergence theorem gives the desired result.

(c) The motivation is the inner function $e^{(z+1)/(z-1)}$ (as mentioned), and the example is constructed and verified from very elementary properties of the Poisson kernel

$$P(r,\theta)=\frac{1-r^2}{1-2r\cos\theta+r^2}=\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right).$$

This well-known fact is the basis for the construction:

$$\int_{-\delta}^{\delta} P(r,\theta)^{1+\epsilon}d\theta\to\infty \quad \text{as } r\to 1^-, \quad \text{for any } \epsilon>0, \delta>0.$$

(Let $f(\theta)=n^{(\epsilon/(1+\epsilon))}$ for $-1/2n<\theta<1/2n$ and 0 otherwise; then

$$\int_{-\delta}^{\delta} P(r,\theta)f(\theta)d\theta\to n^{(\epsilon/(1+\epsilon))} \quad \text{as } r\to 1^-.$$

However,

$$\int Pf\leqq\left(\int P^{1+\epsilon}\right)^{(1/(1+\epsilon))}\left(\int f^{((1+\epsilon)/\epsilon)}\right)^{(\epsilon/(1+\epsilon))}=\left(\int P^{1+\epsilon}\right)^{(1/(1+\epsilon))}.$$

Thus

$$n^{(\epsilon/(1+\epsilon))}\leqq\liminf\left(\int P^{1+\epsilon}\right)^{(1/(1+\epsilon))} \quad\text{or}\quad n^{\epsilon}\leqq\liminf\left(\int P^{1+\epsilon}\right).$$

This, for all n, implies $\liminf\ (\int P^{1+\epsilon})=+\infty$, as desired.)

In particular, $\int_{-\delta}^{\delta}[P(r,\theta)]^2\,d\theta\to\infty$ and hence $\int_{-\delta}^{\delta}\exp\epsilon P(r,\theta)\,d\theta\to\infty$ as $r\to 1^-$, for any $\epsilon>0$, $\delta>0$.

The function ϕ will be $\phi(z)=\prod_{n=1}^{\infty}\exp\epsilon_n((r_nz+1)/(r_nz-1))$ where $\epsilon_n>0$ and $\sum\epsilon_n=1$ and $r_n\to 1^-$ (to be chosen).

Observe that any such ϕ is analytic on $\{|z|\leqq 1, z\neq 1\}$, continuous on $\{|z|\leqq 1\}$ and outer. Indeed, since $(r_nz+1)/(r_nz-1)\to(z+1)/(z-1)$ uniformly (as $n\to\infty$) for all z with $|z-1|\geqq\delta>0$ (for any δ) and $\sum\epsilon_n<\infty$, ϕ is analytic on the complement of $\{1/r_1, 1/r_2,\cdots,1\}$. To prove continuity at $z=1$, first observe that $\log|\phi(re^{i\theta})|=-\sum\epsilon_nP(rr_n,\theta)$, so that $\log|\phi|$ is the Poisson integral of the function $-\sum\epsilon_nP(r_n,\theta)$. This proves ϕ is outer. We will be finished if we show $\sum\epsilon_nP(r_n,\theta)$ is continuous from $-\pi\leqq\theta\leqq\pi$ to $[0,\infty]$. Each $P(r_n,\theta)$ is even and is monotonic on either side of $\theta=0$; the same holds for the sum, since $\epsilon_n>0$. Since $\int\sum d\theta/2\pi=1$, $\sum$ cannot be identically $+\infty$ in any interval; so $\sum$ converges at each θ, except possibly $\theta=0$. Monotonicity gives uniform convergence for $|\theta|\geqq\delta>0$, any δ. Monotonicity and evenness imply that as $\theta\to 0$, $\sum$ approaches either some finite limit or $+\infty$, and it is easy to see that this limit is $\sum(0)$. In any case, the product converges to ϕ uniformly on $|z|\leqq 1$.

Now select any $\epsilon_n > 0$, $\sum \epsilon_n = 1$ (say, $\epsilon_n = 2^{-n}$). Inductively, we will select $\rho_1 < r_1 < \rho_2 < r_2 < \cdots$ and $I_1 \supseteq I_2 \supseteq \cdots$ open intervals containing 0 so that $\rho_n \to 1$, $I_n \downarrow \{0\}$ and the $\phi(z)$ defined above for $\{(\epsilon_n, r_n)\}$ has

$$\int_{-\pi}^{\pi} \left| \frac{\phi(e^{i\theta})}{\phi(\rho_n e^{i\theta})} \right|^{(1/n)} \frac{d\theta}{2\pi} > n.$$

(The purpose of I_n will become apparent.)

Inductively, we may assume ρ_k, r_k, I_k are defined for all $k < n$. (The case $n = 1$ is vacuous.) Let $a_n = \min_\theta \exp - \sum_{k<n} \epsilon_k P(r_k, \theta)$. Then $0 < a_n \leqq 1$ ($a_1 = 1$ since the empty sum $= 0$). For convenience, let $I_0 = (-\pi, \pi)$.

Since $\int \exp \epsilon P(r, \theta) d\theta / 2\pi \to \infty$ as $r \to 1$, no matter what $\epsilon > 0$ is, let ρ_n be chosen between r_{n-1} and 1 and so close to 1 that

$$\int_{I_{n-1}} \exp \frac{1}{n} \epsilon_n P(\rho_n, \theta) \frac{d\theta}{2\pi} > \frac{(n+1)e}{a_n} \cdot$$

Now choose $I_n \subseteq I_{n-1}$ so small that $\int_{I_{n-1}-I_n}$ is still $> (n+\frac{1}{2})e/a_n$. Now we can choose r_n between ρ_n and 1 and so close to 1 that $P(r_n, \theta) < 1$ outside I_n, since $P(r, \theta) \to 0$ uniformly outside I_n, and

$$\int_{I_{n-1}-I_n} \exp \frac{1}{n} \epsilon_n P(r_n \rho_n, \theta) \frac{d\theta}{2\pi} > \frac{ne}{a_n},$$

since $P(r\rho, \theta) \to P(\rho, \theta)$ uniformly as $r \to 1$.

This inductive step defines ρ_n, r_n, I_n for all $n \geqq 1$ and we put $\phi(z) = \prod_1^\infty \exp \epsilon_n (r_n z + 1)/(r_n z - 1)$. It remains to show

$$\int_{-\pi}^{\pi} \left| \frac{\phi(e^{i\theta})}{\phi(\rho_n e^{i\theta})} \right|^{(1/n)} \frac{d\theta}{2\pi} \geqq n.$$

$$\left| \frac{\phi(e^{i\theta})}{\phi(\rho_n e^{i\theta})} \right|^{(1/n)} = \exp \frac{1}{n} \sum_{k=1}^{\infty} \epsilon_k [P(r_k \rho_k, \theta) - P(r_k, \theta)]$$

$$\geqq \exp\left(\frac{1}{n} \sum_1^{n-1}\right) \cdot \exp\left(\frac{1}{n} \epsilon_n P(r_n \rho_n, \theta)\right) \cdot \exp\left(-\frac{1}{n} \sum_n^\infty \epsilon_k P(r_k, \theta)\right),$$

discarding all the positive exponents $P(r_k \rho_k, \theta)$ for $k \geqq n+1$. Examining these three factors on $I_{n-1} - I_n$ we find: The first $\geqq a_n^{(1/n)} \geqq a_n$, by discarding the $P(r_k \rho_n, \theta)$ from the exponent. The third $\geqq \exp(-(1/n) \sum_n^\infty \epsilon_k \cdot 1) > \exp(-1/n) \geqq e^{-1}$, since $I_k \supset I_{k+1} \supset \cdots$ and $P(r_k, \theta) < 1$ outside I_k. The second has $\int_{I_{n-1}-I_n} \cdots d\theta/2\pi > ne/a_n$ by choice. Therefore,

$$\int_{-\pi}^{\pi}\left|\frac{\phi(e^{i\theta})}{\phi(\rho_n e^{i\theta})}\right|^{(1/n)}\frac{d\theta}{2\pi} > \int_{I_{n-1}-I_n}\cdots\frac{d\theta}{2\pi} > a_n\cdot\frac{ne}{a_n}\cdot e^{-1} = n,$$

as claimed.

It is clear that

$$\lim_n \int\left|\frac{\phi(e^{i\theta})}{\phi(\rho_n e^{i\theta})}\right|^{\epsilon} d\theta = +\infty,$$

for any $\epsilon > 0$.

STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305