PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 1, September 1971

CONTINUITY OF SYSTEMS OF DERIVATIONS ON F-ALGEBRAS

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ABSTRACT. Let A be a commutative semisimple F-algebra with identity, and let $D_0, D_1, \cdots$ be a system of derivations from A into the algebra of all continuous functions on the spectrum of A. It is shown that the transformations $D_0, D_1, \cdots$ are necessarily continuous. This result is used to obtain a characterization of derivations on $\mathrm{Hol}(\Omega)$ where Ω is an open polynomially convex subset of C^n.

Introduction. In [4] Gulick proves that if $D_0, D_1, \cdots, D_n$ is a system of derivations from a commutative semisimple regular F-algebra into $C(S(A))$, the algebra of all continuous functions on the spectrum $S(A)$ of A, then the functions $D_0, D_1, \cdots, D_n$ are all continuous. This theorem is not as general as might be hoped for, since the regularity condition is far too restrictive. It precludes, for example, the existence of analytic structure in the algebra, while algebras of analytic functions are among the most interesting algebras which have derivations. It is shown in this paper that the regularity condition is superfluous. We use a lemma proved by Johnson [5] and a technique we developed in [2] to prove the more general theorem that if A is a commutative semisimple F-algebra and $D_0, D_1, \cdots$ is a system of derivations from A into $C(S(A))$, then each of the D_i, $i=0, 1, \cdots$, is necessarily continuous. This theorem allows us to characterize derivations on $\mathrm{Hol}(\Omega)$ for Ω an open polynomially convex subset of C^n.

Preliminaries. An F-algebra is an algebra over the complex numbers which is a complete T_2 topological space with respect to a topology determined by a countable family of multiplicative seminorms $\{\|\cdot\|_i\}$, $i=1, 2, \cdots$. No generality is lost if the seminorms are assumed to be increasing. That is, we may assume $\|\cdot\|_i \leqq \|\cdot\|_{i+1}$ for $i=1, 2, \cdots$. In this paper we will always assume that the algebras under consideration are commutative and have identities. The

Presented to the Society, January 24, 1971 under the title *Systems of derivations and algebras of functions*; received by the editors December 2, 1970.

AMS 1969 *subject classifications*. Primary 4650.

Key words and phrases. F-algebra, derivation, systems of derivations, continuity of derivations.

[1] The author would like to thank the referee for pointing out an error in the original proof of Theorem 5.

spectrum $S(A)$ of an F-algebra A is the space of all continuous homomorphisms of the algebra A onto the complex numbers. $S(A)$ is given the Gelfand topology. If A is an F-algebra, then A can be realized as the inverse limit of a sequence of Banach algebras A_n (see [6]). There is a natural homeomorphism of the spectrum $S(A_n)$ onto a compact subset of $S(A)$. We will henceforth identify $S(A_n)$ with its image in $S(A)$; with this identification, $S(A)$ is the union of the compact sets $S(A_n)$ (see [1], [6]). For an element f of A we will use $f^\wedge$ to denote the Gelfand transform of f. That is, $f^\wedge$ is the function defined on $S(A)$ by $f^\wedge(\phi)=\phi(f)$ for every ϕ in $S(A)$. An F-algebra A is semisimple iff for an element f of A, $\phi(f)=0$ for every $\phi\in S(A)$ implies $f=0$ (see [6]).

We will use the symbol $\mathbf{C}$ to denote the complex numbers. For a topological space T, $C(T)$ will denote the algebra of all complex valued continuous functions on T.

Systems of derivations. For the rest of this paper, A will denote a semisimple commutative F-algebra with identity, and $S(A)$ will denote the spectrum of A. We fix a sequence $A_1, A_2, \cdots$ of Banach algebras such that A is the inverse limit of the algebras A_i. A collection $D_0, D_1, \cdots$ of linear maps from A into $C(S(A))$ is called a system of derivations on A iff $D_0 f=f^\wedge$ for every f in A, and, for any f, g in A,

$$D_k(fg) = \sum_{i=0}^{k} \binom{k}{i} D_i f D_{k-i} g$$

for each $k=1, 2, \cdots$. For a detailed discussion of systems of derivations, the reader is referred to [4].

LEMMA 1 (ROSENFELD [7]). *If $\phi\in S(A)$ is isolated in each $S(A_n)$ which contains it, then ϕ is isolated in $S(A)$.*

LEMMA 2 (JOHNSON [5]). *Let $\phi_1, \phi_2, \cdots$ be a sequence of distinct points in $S(A)$. For each positive integer k, there is a $y_k\in A$ such that $\phi_i(y_k)=0$ for $i<k$ and $\phi_i(y_k)\neq 0$ for $i\geqq k$.*

PROOF. For F-algebras this lemma admits an easier proof than the one originally given by Johnson. The reader is referred to [2] for the proof.

For a point $\phi\in S(A)$, let h_ϕ denote the functional on $C(S(A))$ defined by $h_\phi(f)=f(\phi)$ for every $f\in C(S(A))$.

LEMMA 3. *Suppose that A is a semisimple F-algebra, that ϕ is an* *isolated point of $S(A)$, and that $D_0, D_1, \cdots$ is a system of derivations on A. Then for $i=1, 2, \cdots$ we have that $h_\phi D_i$ is the zero functional on A.*

PROOF. The Shilov idempotent theorem implies there is an $e \in A$ such that $e^\wedge(\psi)=1$ for $\psi \in S(A)-\{\phi\}$ and $e^\wedge(\phi)=0$. Since A is semisimple e is an identity for the ideal $I=\ker \phi$.

The functional $h_\phi D_1$ is a point derivation at ϕ and consequently I^2+C is contained in kernel $(h_\phi D_1)$. Since e is an identity for I, we have $I^2=I$. Therefore $h_\phi D_1=0$.

Fix a positive integer k, and assume that $h_\phi D_1, \cdots, h_\phi D_k$ are zero. For any $f \in I$, we have

$$h_\phi D_{k+1}(f) = h_\phi \sum_{i=0}^{k+1} \binom{k+1}{i} D_i(e) D_{k+1-i}(f) = 0.$$

For an arbitrary $f \in A$, we have $f=g+\alpha$ where $g \in I$ and $\alpha \in \mathbf{C}$. An easy induction proof shows that $D_j(\alpha)=0$ for any $\alpha \in \mathbf{C}$ and $j=1, 2, \cdots$. Therefore $h_\phi D_{k+1}=0$, and induction implies $h_\phi D_1, h_\phi D_2, \cdots$ are all zero. $\square$

Let $y \in A$, $\phi \in S(A)$, and $D_0, D_1, \cdots$ be a system of derivations on A.

LEMMA 4. *Suppose $\phi(y)=0$; fix a positive integer k. Then for any $j<k$ we have $h_\phi D_j y^k=0$.*

PROOF. It is clear that $h_\phi D_0 y=0$.

Fix a positive integer n and suppose that $h_\phi D_j y^n=0$ for $j<n$. Let $j<n+1$. Then

$$h_\phi D_j y^{n+1} = h_\phi \sum_{i=0}^{j} \binom{j}{i} D_i y D_{j-i} y^n = h_\phi(D_0 y D_j y^n) = 0.$$

The lemma now follows from the axiom of induction. $\square$

THEOREM 5. *Let A be a semisimple F-algebra with identity and $D_0, D_1, \cdots$ be a system of derivations from A into $C(S(A))$ such that $D_0 f=f^\wedge$ for every $f \in A$. Then each of the derivations D_i, $i=0, 1, \cdots$, is continuous.*

PROOF. It is clear that D_0 is continuous. Fix a positive integer k, and assume that D_i is continuous for $i<k$. We will use the closed graph theorem to show that D_k is continuous. However, $C(S(A))$ is complete with respect to the compact open topology if and only if $S(A)$ is a k-space, and it has recently been shown that there are F-algebras B for which $S(B)$ is not a k-space (see [3]). In order to

apply the closed graph theorem, we regard D_k not as a function into $C(S(A))$ but rather as a function into $C(k(S(A)))$. Where $k(S(A))$ is the set $S(A)$ with the weak topology generated by the Gelfand compact subsets of $S(A)$. The space $C(k(S(A)))$ is complete with respect to the compact open topology. Hence, we can conclude $D_k : A \to C(k(S(A)))$ is continuous if we can show its graph is closed; this will imply D_k is continuous when regarded as a function into $C(S(A))$.

In order to show that the graph of D_k is closed, we consider the set $T = \{\phi \in S(A) : h_\phi D_k \text{ is continuous}\}$.

Let ϕ_0 be an isolated point in $S(A)$. It follows from Lemma 3 that ϕ_0 is in T. Hence, T contains all of the isolated points of $S(A)$.

Let ϕ_0 be an element of $S(A)$ and suppose ϕ_0 is not isolated in $S(A)$; Lemma 1 implies there is an integer n such that ϕ_0 is not isolated in $S(A_n)$. If ϕ_0 is not a limit point of $S(A_n) \cap T$, then we can choose a sequence $\phi_1, \phi_2, \cdots$ of distinct points in $S(A_n) - T$.

Use Lemma 2 to select a sequence $\{y_i\} \subset A$ such that $\phi_i(y_j) = 0$ for $i < j$ and $\phi_i(y_j) \neq 0$ for $i \geqq j$. Making use of the fact that $h_{\phi_i} D_k$ is not continuous at zero and that D_j is continuous for $j < k$, we choose a sequence $\{x_i\} \subset A$ such that

$$\max\left\{ \|x_i(y_1 \cdots y_i)^{k+1}\|_i,\ \|x_i(y_1 \cdots y_{j-1} y_{j+1} \cdots y_i)^{k+1}\|_i \quad j = 2, \cdots, i-1 \right\} < 2^i$$

and

$$h_{\phi_i} D_k x_i \geqq \left[i + \left| h_{\phi_i} D_k \sum_{j=0}^{i-1} x_j (y_1 \cdots y_j)^{k+1} \right| + \left| h_{\phi_i} \sum_{j=1}^{k} \binom{k}{j} D_j (y_1 \cdots y_i)^{k+1} D_{k-j} x_i \right| \right] |\phi_i(y_1 \cdots y_i)|^{-k-1}.$$

The symbol $\|\cdot\|_i$ denotes the ith seminorm on A, and we assume the seminorms are increasing.

The series $\sum_{i=1}^{\infty} x_i (y_1 \cdots y_i)^{k+1}$ converges to an element x of A. For any positive integer i, we have

$$h_{\phi_i} D_k x = h_{\phi_i} D_k \sum_{j=1}^{i-1} x_j (y_1 \cdots y_j)^{k+1} + h_{\phi_i} D_k x_i (y_1 \cdots y_i)^{k+1} + h_{\phi_i} D_k \sum_{j=i+1}^{\infty} x_j (y_1 \cdots y_j)^{k+1}.$$

The expression $h_{\phi_i} D_k \sum_{j=i+1}^{\infty} x_j (y_1 \cdots y_j)^{k+1}$ may be written as

$$h_{\phi_i}\sum_{l=0}^{k}\binom{k}{l}D_l y_{i+1}^{k+1}D_{k-l}\sum_{j=i+1}^{\infty}x_j(y_1\cdots y_i y_{i+2}\cdots y_j)^{k+1}.$$

Using Lemma 4, we see that this expression is zero. Hence

$$\begin{aligned}|h_{\phi_i}D_kx| = \Bigg| h_{\phi_i}D_k\sum_{j=1}^{i-1}x_j(y_1\cdots y_j)^{k+1} &+ (h_{\phi_i}D_kx_i)\phi_i(y_1\cdots y_i)^{k+1}\\ &+ h_{\phi_i}\sum_{j=1}^{k}\binom{k}{j}D_j(y_1\cdots y_i)^{k+1}D_{k-j}x_i\Bigg| > i.\end{aligned}$$

This shows that the function D_kx is unbounded on $S(A_n)$. But this is a contradiction, since D_kx is a continuous function on $S(A_n)$ and $S(A_n)$ is compact. Hence ϕ_0 must be a limit point of $T\cap S(A_n)$.

We thus have that for each $\phi\in S(A)$ there is an integer n such that ϕ is in the closure of $T\cap S(A_n)$. Since each $f\in C(k(S(A)))$ is continuous when restricted to $S(A_n)$, we have that $\{h_\phi:\phi\in T\}$ separates the points of $C(k(S(A)))$. This is sufficient to guarantee that the graph of D_k is closed. Hence D_k is continuous. Induction implies D_i is continuous for each $i=1, 2, \cdots$. □

By a derivation on the algebra A, we mean a linear function D from A into $C(S(A))$ such that $D(fg)=f^\wedge Dg+g^\wedge Df$ for any $f, g\in A$. Now suppose D is a derivation on A, and $x_1, \cdots, x_n$ are elements of A, and p is a polynomial in n variables. An elementary calculation shows that $Dp(x_1, \cdots, x_n)=\sum_{i=1}^{n}(Dx_i)d_ip(x_1^\wedge, \cdots, x_n^\wedge)$. Here d_i denotes the partial derivative of p with respect to the ith variable.

Consider the algebra $\mathrm{Hol}(\Omega)$ for a polynomially convex open subset Ω of $\mathbf{C}^n$. ($\mathrm{Hol}(\Omega)$ denotes the algebra of all analytic functions on Ω with the compact open topology.) Since Ω is polynomially convex, polynomials in the coordinate functions $z_1, \cdots, z_n$ are dense in $\mathrm{Hol}(\Omega)$. Let D be a derivation from $\mathrm{Hol}(\Omega)$ into $C(\Omega)$. For any polynomial p, we have $Dp(z_1, \cdots, z_n)=\sum_{i=1}^{n}Dz_id_ip(z_1, \cdots, z_n)$. If $f\in\mathrm{Hol}(\Omega)$, we can choose a sequence $\{p_j\}$ of polynomials such that $\{p_j\}$ converges to f with respect to the compact open topology. For each $i=1, \cdots, n$, we have $\{d_ip_j\}_j$ converges to d_if. Theorem 5 implies that D is continuous. Therefore $Df=\sum_{i=1}^{n}Dz_id_if$. We thus have the following theorem.

THEOREM 6. *If Ω is an open polynomially convex subset of $\mathbf{C}^n$ and D is a derivation from* $\mathrm{Hol}(\Omega)$ *into* $C(\Omega)$, *then $Df=\sum Dz_id_if$ for every* $f\in\mathrm{Hol}(\Omega)$.

We note that a similar characterization is valid for systems of derivations on $\mathrm{Hol}(\Omega)$, since Theorem 5 guarantees the continuity of any such system.

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