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PROPERTY L AND DIRECT INTEGRAL DECOMPOSITIONS OF W-* ALGEBRAS

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ABSTRACT. If $\mathfrak{A}$ is a type II_∞ W-* algebra on separable Hilbert space H, $\mathfrak{A}$ is spatially isomorphic to $\mathfrak{B}\otimes B(K)$, $\mathfrak{B}$ of type II_1, K a separable Hilbert space. If $\mathfrak{A}(\lambda)$ are the factors in the direct integral decomposition of $\mathfrak{A}$, the set $\mathfrak{L}=\{\lambda|\mathfrak{A}(\lambda)$ has property $L\}$ is μ-measurable, and $\mathfrak{A}$ has property L iff $\mu(\Lambda-\mathfrak{L})=0$.

Let $\mathfrak{A}$ be a W-* algebra on a separable Hilbert space H; $\mathfrak{A}$ has a direct integral decomposition into factors given by

$$\mathfrak{A}=\int_\Lambda \oplus\ \mathfrak{A}(\lambda)\mu(d\lambda).$$

Our aim is to extend [8, Theorems 4.2 and 4.3] to algebras $\mathfrak{A}$ of type II_∞, i.e. to algebras such that $\mathfrak{A}(\lambda)$ is type II_∞ μ-a.e. We do this by giving a tensor product decomposition for such algebras and applying to this recent work of M. Glaser [5]. The author wishes to thank M. Glaser for making this work available.

We shall use the following notation and general results in this paper (for definitions and proofs see [8]). K denotes the underlying separable Hilbert space of H. For a fixed sequence $\{x_i\}$ dense in the unit ball of K, define metrics d_1 and d_2 on $B(K)$ which induce respectively the strong and weak operator topologies on bounded subsets of $B(K)$ [7, Lemmas I.4.8 and I.4.9]. Let $M(A)=d_1(A,0)$ and $W(A)=d_2(A,0)$. Then a bounded sequence $A_n\in B(K)$ converges strongly (weakly) to 0 iff $M(A_n)\to 0$ ($W(A_n)\to 0$).

Let $[A,B]=AB-BA$, and let $M(A,B)$ denote $M([A,B])$. Let S denote the unit ball of $B(K)$, furnished with the strong-* topology. Finally, let $B_n\in\mathfrak{A}$ be a sequence in the unit ball of $\mathfrak{A}$ such that $B_n(\lambda)$ is strong-* dense in the unit ball of $\mathfrak{A}(\lambda)$ μ-a.e. [8, Lemma 1.5]. We may assume that $B_n(\lambda)$ is strong-* continuous in λ (see remark following [8, Lemma 2.2]).

The following simple lemma is very useful to us.

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LEMMA 1. *Let $A_n \in \mathfrak{A}$ be a bounded sequence. If $A_n \to 0$ strongly (weakly) then there is a subsequence A_{n_k} such that $A_{n_k}(\lambda) \to 0$ strongly (weakly) μ-a.e.*

PROOF. By [7, Lemma I.3.6] and the definition of $M(W)$ we know that $M(A_n(\lambda)) \to 0$ $(W(A_n(\lambda)) \to 0)$ in μ-measure. Hence, by [4, Corollary III.6.13] there is a subsequence A_{n_k} for which this result holds μ-a.e. Q.E.D.

We next introduce a generalization of the definition of central sequence given in [3]. $\mathfrak{Z}$ denotes the center of $\mathfrak{A}$ henceforth.

DEFINITION 2. Let $A_n \in \mathfrak{A}$ be a bounded sequence. A_n is *central* if $[A_n, A] \to 0$ strongly for all $A \in \mathfrak{A}$. A_n is *trivial* if there is a bounded sequence $Z_n \in \mathfrak{Z}$ such that $A_n - Z_n \to 0$ strongly. A_n is *totally nontrivial* if for each subsequence A_{n_k} and each nonzero projection $E \in \mathfrak{Z}$ the sequence $EA_{n_k}E$ is not trivial. If $B_n \in \mathfrak{A}$ is another bounded sequence and $A_n - B_n \to 0$, then we say that A_n and B_n are *equivalent*.

In terms of central sequences we define Pukanszky's property L [6].

DEFINITION 3. $\mathfrak{A}$ has property L if there is a unitary central sequence $U_n \in \mathfrak{A}$ such that $U_n \to 0$ weakly.

A sequence demonstrating property L will be called an L sequence for short. We shall soon show that an L sequence contains a totally nontrivial subsequence; thus property L implies the existence of a totally nontrivial central sequence (t.n.t.c.s.). First we need a preliminary lemma. We use $|\ \ |_2$ henceforth to denote the trace norm on $\mathfrak{A}$ if $\mathfrak{A}$ is type II_1 [8, Definition 2.6].

LEMMA 4. *Let $A_n \in \mathfrak{A}$. If $A_n(\lambda)$ is central μ-a.e., then A_n is central. If $\mathfrak{A}$ is type II_1, then if A_n is central there is a subsequence A_{n_k} such that $A_{n_k}(\lambda)$ is central μ-a.e.*

PROOF. If $A_n(\lambda)$ is central μ-a.e., given $B \in \mathfrak{A}$ we have $M(A_n(\lambda), B(\lambda)) \to 0$ μ-a.e. By [4, Corollary III.6.13] this implies $M(A_n(\lambda), B(\lambda)) \to 0$ in μ-measure, whence $[A_n, B] \to 0$ strongly by [7, Lemma I.3.6]. Hence A_n is central.

If $\mathfrak{A}$ is II_1, note that $A_n(\lambda)$ is central in $\mathfrak{A}(\lambda)$ iff $|[A_n(\lambda), B_j(\lambda)]|_2 \to 0$ for each $j = 1, 2, \cdots$. Since A_n is central, for each j we have $[A_n, B_j] \to 0$ strongly, whence there is a subsequence A_{n_k} such that $M(A_{n_k}(\lambda), B_j(\lambda)) \to 0$ μ-a.e. by Lemma 1 and the Cantor diagonal process. Hence $|[A_{n_k}(\lambda), B_j(\lambda)]|_2 \to 0$ μ-a.e. by [8, Lemma 2.12]. Q.E.D.

LEMMA 5. *An L sequence contains a t.n.t. subsequence.*

PROOF. If $U_n \in \mathfrak{A}$ is an L sequence, we may assume by Lemma 1 that $U_n(\lambda) \to 0$ weakly μ-a.e. Note that if $Z_n \in \mathfrak{Z}$ is a bounded sequence, then by weak compactness we may assume a subsequence (again called Z_n) converges weakly to $Z \in \mathfrak{Z}$; since $Z_n(\lambda) = z_n(\lambda)I$ for a bounded μ-measurable function, it follows that $Z_n \to Z$ strongly. Thus if $E \in \mathfrak{Z}$ is a projection such that $EU_nE - Z_n \to 0$ strongly, we may assume that for all λ such that $E(\lambda) = I$ we have $U_n(\lambda) \to z(\lambda)I$ strongly. Thus for such λ we have $Z(\lambda) = 0$, whence we get a contradiction since the U_n are unitary. Thus $E = 0$. Q.E.D.

Now let $\mathfrak{A}$ be of type II_1 until further comment, and let $A_n \in \mathfrak{A}$ be a fixed bounded sequence. We may assume that $A_n(\lambda)$ is strong-* continuous in λ (see remark following [**8**, Lemma 2.2]).

LEMMA 6. *The set* $N = \{\lambda \mid A_n(\lambda)$ *is trivial*$\}$ *is μ-measurable.*

PROOF. $A_n(\lambda)$ is trivial iff $|A_n(\lambda) - \mathrm{tr}_\lambda(A_n(\lambda))I|_2 \to 0$. If we define, for integers j, k, the sets

$$E(j, k) = \{\lambda \mid |A_n(\lambda) - \mathrm{tr}_\lambda(A_n(\lambda))I|_2 \leqq 1/j,\ n = k, k+1, \cdots\},$$

then $N = \bigcap_{j=1}^\infty \bigcup_{k=1}^\infty \bigcap_{m=k}^\infty E(j, m)$. Each $E(j, k)$ is closed (see [**8**, Lemma 3.4]), whence N is μ-measurable. Q.E.D.

PROPOSITION 7. *$\mathfrak{A}$ has a t.n.t.c.s. iff $\mathfrak{A}(\lambda)$ has a t.n.t.c.s. μ-a.e.*

PROOF. If $\mathfrak{A}(\lambda)$ has a t.n.t.c.s. μ-a.e. then $\mathfrak{A}(\lambda)$ has property Γ [**2**, Proposition 1.10] and hence property L [**9**, Theorem 3] μ-a.e. By [**8**, Theorem 4.3] $\mathfrak{A}$ has property L, and the result follows from Lemma 5. Conversely, if $A_n \in \mathfrak{A}$ is a t.n.t.c.s., we may assume by Lemma 4 that $A_n(\lambda)$ is central μ-a.e. If $\mu(N) > 0$, where N is as in Lemma 6, then $E \in \mathfrak{Z}$ such that $E(\lambda) = f(\lambda)I$, f the characteristic function of N, is a projection such that $E \neq 0$ and EA_nE is trivial. Hence $\mu(N) = 0$ and the result follows. Q.E.D.

COROLLARY 8. *$\mathfrak{A}$ has a t.n.t.c.s. iff $\mathfrak{A}$ has property L.*

We next turn our attention to algebras of type II_∞. We begin by proving a structure theorem patterned on [**7**, Theorem II.2.22].

THEOREM 9. *If $\mathfrak{A}$ is type II_∞, then $\mathfrak{A}$ is spatially isomorphic to $\mathfrak{B} \otimes B(J)$, where $\mathfrak{B}$ is type II_1 and J is a separable Hilbert space.*

PROOF. It suffices to demonstrate the existence of a sequence of equivalent, finite, mutually orthogonal projections $E_n \in \mathfrak{A}$ such that $I = \sum_{n=1}^\infty E_n$ and to apply the argument of [**7**, pp. 104–105]. We produce this sequence as follows. Let S^∞ denote the Cartesian product of countably many copies of S. S^∞ is a separable metric space with respect to the product metric arising from the strong-* topology. Let $T=\{x\in K \mid |x|=1\}$. For positive integers j, k, define the subset $E(j, k)$ of $\Lambda\times S^\infty\times S^\infty\times T$ as the set of (λ, F_n, V_n, x) satisfying the following conditions:

(1) $F_n, V_n\in\mathfrak{A}(\lambda), n=1, 2, \cdots$.

(2) $F_n=F_n^2=F_n^*, n=1, 2, \cdots$.

(3) $F_1=V_nV_n^*, F_n=V_n^*V_n, n=1, 2, \cdots$.

(4) $F_nF_m=F_mF_n=0, m\neq n, m, n=1, 2, \cdots$.

(5) $F_1x=x$.

(6) $((F_1B_m(\lambda)F_1)(F_1B_n(\lambda)F_1)x, x)=((F_1B_n(\lambda)F_1)(F_1B_m(\lambda)F_1)x, x)$, $m, n=1, 2, \cdots$.

(7) $M(I-\sum_{i=1}^k F_i)\leqq 1/j$.

Let $R=\bigcap_{j=1}^\infty\bigcup_{k=1}^\infty E(j, k)$, and let π denote the projection of $\Lambda\times S^\infty\times S^\infty\times T$ onto Λ. It follows from [**7**, Lemma III.1.7] and the known structure of type II_∞ factors that Λ differs by a μ-null set from $\pi(R)$. Hence by [**7**, Lemma I.4.7] there exist μ-measurable functions $F_n(\lambda)$, $V_n(\lambda)$, and $x(\lambda)$ defined μ-a.e. such that $(\lambda, F_n(\lambda), V_n(\lambda), x(\lambda))\in R$ μ-a.e. If we put $E_n=\int_\Lambda\oplus F_n(\lambda)\mu(d\lambda)$ and recall that $F_1(\lambda)$ (and hence each $F_n(\lambda)$) is finite by [**7**, Lemma III.1.7], and that $\sum_{n=1}^\infty E_n=I$ by [**7**, Lemma I.3.6], we are done. Q.E.D.

We now apply Theorem 9 to extend [**8**, Theorems 4.2 and 4.3] to $\mathfrak{A}$ of type II_∞. We remark that [**8**, Theorem 4.3] should be stated as "if and only if", and we use this in our proof.

THEOREM 10. *Let $\mathfrak{A}$ be of type* II_∞. *Then $\mathfrak{L}=\{\lambda\mid\mathfrak{A}(\lambda)$ has property $L\}$ is μ-measurable, and $\mathfrak{A}$ has property L iff $\mu(\mathfrak{L}')=0$, where $\mathfrak{L}'=\Lambda-\mathfrak{L}$.*

PROOF. By Theorem 9, $\mathfrak{A}$ is spatially isomorphic to $\mathfrak{B}\otimes B(J)$, where $\mathfrak{B}$ is of type II_1. By [**1**, Proposition II.3.3] and [**7**, Theorem I.6.1] we may assume that $\mathfrak{A}(\lambda)=\mathfrak{B}(\lambda)\otimes B(J)$, where $\mathfrak{B}=\int_\Lambda\oplus B(\lambda)\mu(d\lambda)$.

Note that a bounded sequence equivalent to t.n.t.c.s. is also t.n.t.c.s. Glaser has proved that if $A_n\in\mathfrak{B}(\lambda)\otimes B(J)$ is central, then A_n is equivalent to a sequence $B_n\otimes I$ such that $|B_n|\leqq|A_n|$ [**5**, Lemma 4.3]. Conversely, if $B_n\in\mathfrak{B}(\lambda)$ is central, then $B_n\otimes I$ is central [**5**, Theorem 5.5]. By these results, Lemma 5, and the initial part of the proof of Proposition 7, we see that $\mathfrak{L}=\{\lambda\mid\mathfrak{B}(\lambda)$ has property $L\}$, whose measurability is proved in [**8**, Theorem 4.2].

Suppose next that $\mathfrak{A}$ has property L. Then $\mathfrak{A}$ and therefore $\mathfrak{B}$ has a t.n.t.c.s. [**5**, Lemma 4.3]. Hence $\mu(\mathfrak{L}')=0$ by Corollary 8 and [**8**, Theorem 4.3]. Conversely, suppose $\mu(\mathfrak{L}')=0$. Then $\mathfrak{A}(\lambda)$ and hence $\mathfrak{B}(\lambda)$ has a t.n.t.c.s. μ-a.e. Hence $\mathfrak{B}$ has property L, and so by Lemma 4 there is an L sequence $U_n\in\mathfrak{B}$ such that $U_n(\lambda)$ is central in $\mathfrak{B}(\lambda)$

μ-a.e. Hence, $U_n \otimes I$ is central in $\mathfrak{A}$ by the result of Glaser and Lemma 4. It is then clear that $\mathfrak{A}$ has property L. Q.E.D.

We remark in conclusion that, except for [8, Theorems 4.6 and 4.7], there are no results of this type known for type III algebras.

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