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THE CONJUGATION REPRESENTATION AND FUSIONLESS EXTENSIONS

EDWARD FORMANEK

ABSTRACT. Let G be a finite group with center Z. G acts on itself by conjugation, and this induces a permutation representation P of G/Z. It has been conjectured that every complex irreducible representation of G/Z occurs in P. This paper gives a counterexample to this conjecture.

Let G be a finite group with center Z. G acts on itself by conjugation and this action yields a permutation representation P of G. P is in fact a representation of G/Z, and this motivates the following question, asked by Richard Roth [**2**]:

Does every complex irreducible representation of G/Z occur in P?

The answer is no, and the purpose of this paper is to give a counterexample.

LEMMA 1. *Suppose R is a one-dimensional representation of G with kernel K. Then R occurs in P iff K contains the centralizer of some element of G.*

PROOF. Let $\{g_1\}, \cdots, \{g_n\}$ = conjugacy classes of G; H_i = centralizer of g_i; P_i = permutation representation of G afforded by the action of G on $\{g_i\}$ = representation of G induced by the principal representation X_i of H_i.

$$P = P_1 + \cdots + P_n \quad \text{and} \quad (R \mid H_i, X_i)_{H_i} = (R, P_i)_G$$

by Frobenius reciprocity. Thus R occurs in P (and hence in some P_i) iff K contains some H_i (i.e. $R \mid H_i = X_i$).

Suppose that ϕ is an outer automorphism of a group K and that ϕ has order r as an outer automorphism (i.e. ϕ^r is inner, but no smaller power of ϕ is inner). If ϕ^r acts like conjugation by a in K, we will let (K, ϕ, a) denote the group $\{K, x : x^r = a, k^x = \phi(k) \text{ for all } k \in K\}$.

LEMMA 2. *Suppose that $G = (K, \phi, a)$ is defined as above, and that ϕ carries each conjugacy class of K into itself (i.e. G is a fusionless extension of K). Let R be a one-dimensional complex representation of G with kernel K. Then R is a representation of G/Z, but R does not occur in the conjugation representation P of G.*

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PROOF. K contains the center Z of G since ϕ is outer. K does not contain the centralizer of any element of G since ϕ leaves invariant the conjugacy classes of K. Thus R does not occur in P, by Lemma 1.

By virtue of Lemma 2 we need only observe that there are finite groups with class preserving outer automorphisms. The first examples were given by Burnside [1].

REFERENCES

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UNIVERSITY OF MISSOURI-ST. LOUIS, ST. LOUIS, MISSOURI 63121