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CONCERNING DIAGONAL SIMILARITY OF IRREDUCIBLE MATRICES

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ABSTRACT. If $A=(a_{ij})$ is an $n\times n$ irreducible matrix, then there are positive numbers $d_1, d_2, \cdots, d_n$ so that $\sum_k d_i a_{ik} d_k^{-1} = \sum_k d_k a_{ki} d_i^{-1}$ for each $i\in\{1, 2, \cdots, n\}$. Further, the numbers $d_1, d_2, \cdots, d_n$ are unique up to scalar multiples.

Introduction. Sinkhorn and Knopp in [4] as well as Brualdi, Parter, and Schneider in [1] have shown that if A is a fully indecomposable matrix then there are diagonal matrices D_1 and D_2 with positive main diagonals so that D_1AD_2 is doubly stochastic. Further, in each paper D_1 and D_2 are shown to be unique up to scalar multiples.

In this paper we prove what is considered the analogue of the above result in terms of irreducible matrices. That is we show that if A is an $n\times n$ irreducible matrix then there is a diagonal matrix D, with positive main diagonal, so that

$$\sum_k d_i a_{ik} d_k^{-1} = \sum_k d_k a_{ki} d_i^{-1}$$

for each $i\in\{1, 2, \cdots, n\}$. Further we show that D is unique up to scalar multiples.

Definitions and notation. Let $n\geqq 2$ be an integer. Let $N=\{1, 2, \cdots, n\}$. An $n\times n$ nonnegative matrix A is said to be reducible if there is a permutation matrix P so that

$$PAP^T = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where A_1 and A_2 are square. If A is not reducible we say that A is irreducible. By agreement each 1×1 matrix is irreducible. Denote

$$m(A) = \min_{\phi\neq M\not\subseteq N} \sum_{i\in M;\, j\notin M} a_{ij}, \qquad M(A) = \sum_{i\in N;\, j\in N} a_{ij}.$$

Fiedler [2] refers to $m(A)$ as a measure of the irreducibility of A. It should be clear that A is irreducible if and only if $m(A)>0$. Let

$$\Lambda_n = \left\{ n\times n \text{ nonnegative matrices } A \text{ with } \sum_k a_{ik} = \sum_k a_{ki} \text{ for each } i\in N \right\}$$

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and

$$\mathfrak{D} = \Big\{(d_1, d_2, \cdots, d_n) \text{ so that } d_k > 0 \text{ for each } k \in N, \text{ and at least one } d_k = 1\Big\}.$$

For A irreducible we denote by $S(A)$ a positive number so that $S(A)\cdot m(A) - M(A) > M(A)$. Further we define

$$f_A(d_1, d_2, \cdots, d_n) = \max_{\phi \neq M \nsubseteq N} \left| \sum_{i\in M;\, j\in N} d_i a_{ij} d_j^{-1} - \sum_{i\in N;\, j\in M} d_i a_{ij} d_j^{-1} \right|$$
$$= \max_{\phi \neq M \nsubseteq N} \left| \sum_{i\in M;\, j\notin M} d_i a_{ij} d_j^{-1} - \sum_{i\notin M;\, j\in M} d_i a_{ij} d_j^{-1} \right|$$

where each $d_k > 0$, $k \in N$.

Results.

LEMMA 1. *If* $(d_1, d_2, \cdots, d_n) \in \mathfrak{D}$ *and* $\max_{k\in N} d_k \geqq S^n(A)$, *then*

$$f_A(d_1, d_2, \cdots, d_n) > f_A(1, 1, \cdots, 1).$$

PROOF. Reorder $(d_1, d_2, \cdots, d_n)$ to $(d_1', d_2', \cdots, d_n')$ where $d_1' \geqq d_2' \geqq \cdots \geqq d_n'$. Let s denote the smallest positive integer such that $d_s'/d_{s+1}' > S(A)$. That there is such an s follows since if for each $k \in \{1, \cdots, n-1\}$, we have that $d_k'/d_{k+1}' \leqq S(A)$, then $d_1'/d_n' = \prod_{k=1}^{n-1} (d_k'/d_{k+1}') \leqq S^{n-1}(A)$. But $d_1' \geqq \cdots \geqq d_n'$, and *some* $d_k' = 1$ implies $d_n' \leqq 1$. Hence $d_1' \leqq S^{n-1}(A) d_n' \leqq S^{n-1}(A)$. Since $S(A) > 2M(A)/m(A) \geqq 2$, $S^{n-1}(A) < S^n(A)$. It would therefore follow that $\max_k d_k = d_1' < S^n(A)$, a contradiction. Let $M = \{i_R \mid d_{i_R} = d_R' \text{ for each } R \in \{1, 2, \cdots, s\}\}$. Then

$$\begin{aligned} f_A(1, 1, \cdots, 1) &< M(A) < S(A)\cdot m(A) - M(A) \\ &\leqq (d_s')(d_{s+1}')^{-1} \sum_{i\in M;\, j\notin M} a_{ij} - \sum_{i\notin M;\, j\in M} a_{ij} \\ &\leqq \sum_{i\in M;\, j\notin M} d_i a_{ij} d_j^{-1} - \sum_{i\notin M;\, j\in M} d_i a_{ij} d_j^{-1} \leqq f_A(d_1, d_2, \cdots, d_n). \end{aligned}$$

LEMMA 2. *If* $(d_1, d_2, \cdots, d_n) \in \mathfrak{D}$ *and* $\min_{k\in N} d_k \leqq S^{-n}(A)$ *then*

$$f_A(d_1, d_2, \cdots, d_n) > f_A(1, 1, \cdots, 1).$$

PROOF. If for each $k \in \{1, \cdots, n-1\}$ we have that $d_k'/d_{k+1}' \leqq S(A)$, it would follow as in the proof of Lemma 1 that $\min_k d_k = d_n' > S^{-n}(A)$. Hence for this case there is also a smallest integer s such that $d_s'/d_{s+1}' > S(A)$. The rest of the proof is identical with that of Lemma 1.

LEMMA 3. *$f_A(\lambda d_1, \lambda d_2, \cdots, \lambda d_n) = f_A(d_1, d_2, \cdots, d_n)$ for $\lambda > 0$.*

LEMMA 4. *$f_A(d_1, d_2, \cdots, d_n)$ achieves a minimum for some $(d_1, d_2, \cdots, d_n) \in \mathfrak{D}$.*

PROOF. Since $S(A) > 2$, $(1, 1, \cdots, 1)$ belongs to the compact set $\mathcal{K} = \{(d_1, \cdots, d_n) \in \mathfrak{D} \mid S^{-n}(A) \leqq \min_k d_k \leqq \max_k d_k \leqq S^n(A)\}$. By Lemmas 1 and 2, $\mathrm{Inf}_{\mathfrak{D}} f_A = \mathrm{Inf}_{\mathcal{K}} f_A$, and the result follows.

LEMMA 5. *Let $(d_1, d_2, \cdots, d_n) \in \mathfrak{D}$. Suppose*

$$f_A(d_1, d_2, \cdots, d_n) = \max_{\phi \neq M \subsetneq N} \left| \sum_{i \in M;\, j \notin M} d_i a_{ij} d_j^{-1} - \sum_{i \notin M;\, j \in M} d_i a_{ij} d_j^{-1} \right|$$

is achieved on $M_0 = \{i_1, i_2, \cdots, i_s\}$. Then

$$\left\{ \sum_k d_i a_{ik} d_k^{-1} - \sum_k d_k a_{ki} d_i^{-1} \,\middle|\, i \in M_0 \right\}$$

contains no sign changes and $\{\sum_k d_i a_{ik} d_k^{-1} - \sum_k d_k a_{ki} d_i^{-1} \mid i \notin M_0\}$ contains no sign changes.

PROOF. Let $M_0' = \{i \in M_0 \mid \sum_k d_i a_{ik} d_k^{-1} - \sum_k d_k a_{ki} d_i^{-1} > 0\}$ and $M_0'' = \{i \in M_0 \mid \sum_k d_i a_{ik} d_k^{-1} - \sum_k d_k a_{ki} d_i^{-1} < 0\}$. If both M_0' and M_0'' are nonvoid, then we would have

$$\left| \sum_{i \in M_0;\, k \in N} d_i a_{ik} d_k^{-1} - \sum_{i \in M_0;\, k \in N} d_k a_{ki} d_i^{-1} \right|$$

$$< \max\left[\sum_{i \in M_0';\, k \in N} d_i a_{ik} d_k^{-1} - \sum_{i \in M_0';\, k \in N} d_k a_{ki} d_i^{-1}, \right.$$

$$\left. \sum_{i \in M_0'';\, k \in N} d_k a_{ki} d_i^{-1} - \sum_{i \in M_0'';\, k \in N} d_i a_{ik} d_k^{-1} \right],$$

a contradiction. The proof of the other half of the lemma is similar.

THEOREM 1. *If A is irreducible, then there is a*

$$D = \text{diagonal } (d_1, d_2, \cdots, d_n)$$

with $(d_1, d_2, \cdots, d_n) \in \mathfrak{D}$ so that $DAD^{-1} \in \Lambda_n$.

PROOF. We first prove the theorem for a positive matrix A. Suppose $f_A(d_1, d_2, \cdots, d_n)$ achieves a minimum at $(d_1^0, d_2^0, \cdots, d_n^0) \in \mathfrak{D}$. Suppose

$$f_A(d_1^0, d_2^0, \cdots, d_n^0) = \left| \sum_{i\in M_0; j\in N} d_i^0 a_{ij}(d_j^0)^{-1} - \sum_{i\in N; j\in M_0} d_i^0 a_{ij}(d_j^0)^{-1} \right| .$$

We shall prove that $f_A(d_1^0, \cdots, d_n^0) = 0$.

Then $\sum_{i\in M; j\in N} d_i^0 a_{ij}(d_j^0)^{-1} - \sum_{i\in N; j\in M} d_i^0 a_{ij}(d_j^0)^{-1} = 0$ for any M where $\phi \neq M \nsubseteq N$. The result follows by taking $M = \{k\}$, for each $k \in \{1, \cdots, n\}$. Hence suppose that $f_A(d_1^0, \cdots, d_n^0) > 0$. Without loss of generality suppose $d_i^0 a_{ij}(d_j^0)^{-1} = b_{ij}$ and $\sum_{k\in N} b_{ik} - \sum_{k\in N} b_{ki} > 0$ for each $i \in M_0$. Let $M_1 = \{i \mid \sum_{k\in N} b_{ik} = \sum_{k\in N} b_{ki}\}$ and $M_2 = N - (M_0 \cup M_1)$.

In view of Lemma 5, it must be that $\sum_k b_{ik} - \sum_k b_{ki} < 0$ for all $i \in M_2$. In particular, M_2 is nonempty.

Consider $(d_1, d_2, \cdots, d_n)$ defined as follows:

$$\begin{aligned} d_k &= (1+\epsilon)^{-1} && \text{if } k \in M_0, \\ &= 1 && \text{if } k \in M_1, \\ &= (1+\epsilon) && \text{if } k \in M_2, \quad \text{where } 0 < \epsilon < 1. \end{aligned}$$

Now suppose $M_1' \subseteq M_1$. Then consider

$$\begin{aligned} g(\epsilon) = &\sum_{i\in M_0\cup M_1'; j\in N} d_i b_{ij} d_j^{-1} - \sum_{i\in N; j\in M_0\cup M_1'} d_i b_{ij} d_j^{-1} \\ = &\sum_{i\in M_0; j\in M_1} (1+\epsilon)^{-1} b_{ij} + \sum_{i\in M_0; j\in M_2} (1+\epsilon)^{-2} b_{ij} - \sum_{i\in M_1; j\in M_0} (1+\epsilon) b_{ij} \\ &- \sum_{i\in M_2; j\in M_0} (1+\epsilon)^2 b_{ij} + \sum_{i\in M_1'; j\in M_0} (1+\epsilon) b_{ij} + \sum_{i\in M_1'; j\in M_1} b_{ij} \\ &+ \sum_{i\in M_1'; j\in M_2} (1+\epsilon)^{-1} b_{ij} - \sum_{i\in M_0; j\in M_1'} (1+\epsilon)^{-1} b_{ij} \\ &- \sum_{i\in M_1; j\in M_1'} b_{ij} - \sum_{i\in M_2; j\in M_1'} (1+\epsilon) b_{ij}. \end{aligned}$$

Hence $g'(0) < 0$. Therefore there is a number ϵ_1 so that, if $0 < \epsilon < \epsilon_1$,

$$\max_{M_1'} \left(\sum_{i\in M_0\cup M_1'; k\in N} d_i b_{ik} d_k^{-1} - \sum_{i\in M_0\cup M_1'; k\in N} d_k b_{ki} d_i^{-1} \right) < f_A(d_1^0, d_2^0, \cdots, d_n^0)$$

and

$$\sum_{k\in N} d_i b_{ik} d_k^{-1} - \sum_{k\in N} d_k b_{ki} d_i^{-1} > 0 \qquad \text{for } i \in M_0.$$

Similarly, for $M_1'' \subseteq M_1$, consider

$$
\begin{aligned}
h(\epsilon) = & -\sum_{i\in M_2\cup M_1'';\, j\in N} d_i b_{ij} d_j^{-1} + \sum_{i\in N;\, j\in M_2\cup M_1''} d_i b_{ij} d_j^{-1} \\
= & \sum_{i\in M_0;\, j\in M_2} (1+\epsilon)^{-2} b_{ij} + \sum_{i\in M_1;\, j\in M_2} (1+\epsilon)^{-1} b_{ij} \\
& - \sum_{i\in M_2;\, j\in M_0} (1+\epsilon)^2 b_{ij} - \sum_{i\in M_2;\, j\in M_1} (1+\epsilon) b_{ij} \\
& + \sum_{i\in M_0;\, j\in M_1''} (1+\epsilon)^{-1} b_{ij} + \sum_{i\in M_1;\, j\in M_1''} b_{ij} \\
& + \sum_{i\in M_2;\, j\in M_1''} (1+\epsilon) b_{ij} - \sum_{i\in M_1'';\, j\in M_0} (1+\epsilon) b_{ij} \\
& - \sum_{i\in M_1'';\, j\in M_1} b_{ij} - \sum_{i\in M_1'';\, j\in M_2} (1+\epsilon)^{-1} b_{ij}.
\end{aligned}
$$

Hence $h'(0) < 0$. Therefore there is a number ϵ_2 so that, if $0 < \epsilon < \epsilon_2$,

$$
\max_{M_1''} \left(\sum_{i\in M_2\cup M_1'';\, k\in N} d_k b_{ki} d_i^{-1} - \sum_{i\in M_2\cup M_1'';\, k\in N} d_i b_{ik} d_k^{-1} \right) < f_A(\overset{0}{d}_1, \overset{0}{d}_2, \cdots, \overset{0}{d}_n)
$$

and

$$
\sum_{k\in N} d_k b_{ki} d_i^{-1} - \sum_{k\in N} d_i b_{ik} d_k^{-1} > 0 \qquad \text{for } i \in M_2.
$$

Therefore for $\epsilon_3 = \min(\epsilon_1, \epsilon_2)$ and $0 < \epsilon < \epsilon_3$ we have in view of Lemma 5, any set $\overline{M}$ for which

$$
f_A(d_1, \cdots, d_n) = \left| \sum_{i\in \overline{M};\, j\in N} d_i a_{ij} d_j^{-1} - \sum_{i\in N;\, j\in \overline{M}} d_i a_{ij} d_j^{-1} \right|
$$

must necessarily be of one of the two forms $M_0 \cup M_1'$ or $M_2 \cup M_1''$ for some $M_1' \subseteq M_1$ or $M_1'' \subseteq M_1$. Therefore

$$
f_A(d_1\overset{0}{d}_1, d_2\overset{0}{d}_2, \cdots, d_n\overset{0}{d}_n) < f_A(\overset{0}{d}_1, \overset{0}{d}_2, \cdots, \overset{0}{d}_n).
$$

But $(1, d_1^{-1}(d_1^0)^{-1} d_2 d_2^0, \cdots, d_1^{-1}(d_1^0)^{-1} d_n d_n^0) \in \mathfrak{D}$, and since, by Lemma 3,

$$
f(1, d_1^{-1}(\overset{0}{d}_1)^{-1} d_2\overset{0}{d}_2, \cdots, d_1^{-1}(\overset{0}{d}_1)^{-1} d_n\overset{0}{d}_n) = f(d_1\overset{0}{d}_1, d_2\overset{0}{d}_2, \cdots, d_n\overset{0}{d}_n),
$$

f_A would not be minimal at $(d_1^0, \cdots, d_n^0)$. This contradiction shows that $f_A(d_1^0, \cdots, d_n^0) = 0$ as was to be proven.

Now suppose A is irreducible. Let A_m be a positive matrix for each positive integer m, and $\lim_{m\to\infty} A_m = A$. Then for each m there is a diagonal matrix $D_m \in \mathfrak{D}$ so that $D_m A_m D_m^{-1} \in \Lambda_n$. In the proof of Lemma 4, $S^{-n}(A_m) \leqq \min_k d_k^{(m)} \leqq \max_k d_k^{(m)} \leqq S^n(A_m)$. Prudent choices for each

$S(A_m)$ indicate that the $d_k^{(m)}$ are bounded from above and away from zero as $m \to \infty$. Hence $\{D_m\}$ has a limit point D and D is nonsingular. Let $\{D_{m'}\}$ be a subsequence of $\{D_m\}$ so that $\lim_{m' \to \infty} D_{m'} = D$. Then $\lim_{m' \to \infty} D_{m'} A_{m'} D_{m'}^{-1} = DAD^{-1}$ and the result follows.

COROLLARY 1. *If for some permutation matrix P, $P^T AP$ is a direct sum of irreducible matrices then there is a D=*diagonal *$(d_1, d_2, \cdots, d_n)$ with $(d_1, d_2, \cdots, d_n) \in \mathfrak{D}$ such that $DAD^{-1} \in \Lambda_n$.*

We might remark here that the result essentially characterizes irreducibility, in the sense that if A is a nonnegative $n \times n$ matrix then there exists a $D = \text{diag}(d_1, \cdots, d_n)$, $(d_1, \cdots, d_n) \in \mathfrak{D}$ and $DAD^{-1} \in \Lambda_n$, if and only if there exists a permutation matrix P such that $P^T AP$ is a direct sum of irreducible matrices. The converse part of this statement follows readily from the normal form of a reducible matrix [**3**, p. 74].

THEOREM 2. *If A is irreducible, $A \in \Lambda_n$, $B \in \Lambda_n$ and D is a diagonal matrix with positive main diagonal so that $DAD^{-1} = B$, then $A = B$ and $D = \lambda I$ for some $\lambda > 0$.*

PROOF. Without loss of generality we may assume

$$D = \text{diagonal } (d_1, d_2, \cdots, d_n) \qquad \text{where } d_1 \geqq d_2 \geqq \cdots \geqq d_n.$$

If $d_1 = d_n$ then we are through. If $d_1 \neq d_n$, then let s denote the smallest integer so that $d_s > d_{s+1}$. Let $M = \{1, 2, \cdots, s\}$. If $i \in M$, $k \notin M$, $d_i > d_k$ and so $d_i a_{ik} d_k^{-1} \geqq a_{ik}$, the last inequality is strict for some $i \in M$, $k \notin M$ due to the irreducibility of A. Thus

$$\sum_{i \in M; k \notin M} d_i a_{ik} d_k^{-1} > \sum_{i \in M; k \notin M} a_{ik} = \sum_{i \notin M; k \in M} a_{ik}.$$

Similarly $\sum_{i \notin M; k \in M} a_{ik} > \sum_{i \notin M; k \in M} d_i a_{ik} d_k^{-1}$ and, therefore, $\sum_{i \in M; k \notin M} d_i a_{ik} d_k^{-1} > \sum_{i \notin M; k \in M} d_i a_{ik} d_k^{-1}$, i.e.

$$\sum_{i \in M; k \in N} d_i a_{ik} d_k^{-1} > \sum_{i \in N; k \in M} d_i a_{ik} d_k^{-1}$$

which gives us a contradiction. Therefore $d_1 = d_n$ and $D = \lambda I$ for some $\lambda > 0$.

COROLLARY 2. *If A is irreducible and D a diagonal matrix with $DAD^{-1} \in \Lambda_n$, then D is unique up to a scalar multiple.*

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