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ON THE COEFFICIENTS OF BAZILEVIČ FUNCTIONS

F. R. KEOGH[1] AND SANFORD S. MILLER

ABSTRACT. Let $B(\alpha)$ denote the class of normalized ($f(0)=0$, $f'(0)=1$) Bazilevič functions of type α defined in $\Delta:|z|<1$, i.e. $f(z) = [\alpha\int_0^z P(\zeta)g(\zeta)^\alpha\zeta^{-1}d\zeta]^{1/\alpha}$ where $g(\zeta)$ is starlike in Δ, $P(\zeta)$ is regular with Re $P(\zeta)>0$ in Δ and $\alpha>0$. Let $B_m(\alpha)$ denote the subclass of $B(\alpha)$ which is m-fold symmetric ($f(e^{2\pi i/m}z)=e^{2\pi i/m}f(z)$, $m=1, 2, \cdots$). Functions in $B(\alpha)$ have been shown to be univalent. The authors obtain sharp coefficient inequalities for functions in $B_m(1/N)$ where N is a positive integer. In addition an example of a Bazilevič function which is not close-to-convex is given.

Let S denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are univalent in the unit disk $\Delta:|z|<1$. If $g(z)$ is starlike in Δ, $P(z)$ is regular with Re $P(z)>0$ in Δ and $\alpha>0$, then the function

$$f(z) = \left[\alpha \int_0^z P(\zeta)g(\zeta)^\alpha\zeta^{-1}d\zeta\right]^{1/\alpha} \tag{1}$$

has been shown by Bazilevič [1] (see also Pommerenke [6]) to be a regular and univalent function in Δ. The powers appearing in the formula are meant as principal values. We shall call a function in S satisfying (1) a Bazilevič function of type α and denote the class of such functions by $B(\alpha)$. Note that $B(1)$ is the class of normalized close-to-convex functions.

A function $f(z)$ analytic in Δ is said to be m-fold symmetric ($m=2, 3, \cdots$) if $f(e^{2\pi i/m}z)=e^{2\pi i/m}f(z)$. In particular, every odd $f(z)$ is 2-fold symmetric. Let S_m denote the subclass of S consisting of those $f(z)$ that are m-fold symmetric. We similarly define $B_m(\alpha)$. A simple argument shows that $f\in S_m$ is characterized by having a power series of the form $f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots$.

The Bieberbach conjecture remains unsettled for functions in $B(\alpha)$ except for the case $\alpha=1/N$, where N is a positive integer (Zamorski

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[7]). C. Pommerenke [5] has obtained sharp coefficient inequalities for functions in $B_m(1)$. In this paper we shall be concerned mainly with obtaining sharp coefficient inequalities for functions in $B_m(1/N)$. The method is different from the methods of Zamorski and Pommerenke and our results include theirs.

Many of the properties of Bazilevič functions of type α, $0<\alpha<1$, coincide with properties of close-to-convex functions, including some of the results of this paper. We will show, at the end of the paper, an example of a Bazilevič function which is not close-to-convex.

The notation $g(z)\ll h(z)$ ("$g(z)$ is majorized by $f(z)$") will mean that if $g(z)=\sum_0^\infty b_n z^n$ and $h(z)=\sum_0^\infty c_n z^n$ then $|b_n|\leqq c_n$ for $n=0, 1, \cdots$ (see, for example, [3, p. 69]).

We shall need the following lemmas, the first of which is well known.

LEMMA 1. (a) $\phi\in S_m$ *if and only if* $\phi(z)=[f(z^m)]^{1/m}$ *where* $f\in S$. (b) $\phi(z)$ *is m-fold symmetric and starlike if and only if* $\phi(z)=[f(z^m)]^{1/m}$ *where* $f(z)$ *is starlike.*

LEMMA 2. *If* $f(z)$ *is m-fold symmetric and starlike and* $\gamma>0$ *then*

$$[f(z)/z]^\gamma \ll |f'(0)|^\gamma (1-z^m)^{-2\gamma/m}.$$

PROOF. A function $f(z)$ which is m-fold symmetric and starlike has a Herglotz representation given by

$$\log\frac{f(z)}{zf'(0)}=\int_0^{2\pi}\log\frac{1}{(1-z^m e^{-i\phi})^{2/m}}\,d\mu(\phi),$$

where $\mu(\phi)$ is nondecreasing on $[0, 2\pi]$ and $\mu(2\pi)-\mu(0)=1$. Thus

$$\begin{aligned}\log\frac{f(z)}{zf'(0)}&=\frac{2}{m}\int_0^{2\pi}\sum_{n=1}^\infty\frac{z^{mn}e^{-in\phi}}{n}\,d\mu(\phi)\\&=\frac{2}{m}\sum_{n=1}^\infty\left[\int_0^{2\pi}e^{-in\phi}d\mu(\phi)\right]\frac{z^{mn}}{n}\ll\frac{2}{m}\sum_{n=1}^\infty\frac{z^{mn}}{n},\end{aligned}$$

and consequently

$$\log(f(z)/zf'(0))\ll\log(1-z^m)^{-2/m}.$$

If $\gamma>0$, then

$$\gamma\log(f(z)/zf'(0))\ll\gamma\log(1-z^m)^{-2/m},$$

or

$$\log[f(z)/zf'(0)]^\gamma\ll\log(1-z^m)^{-2\gamma/m}.$$

Since exponentiation preserves the majorization property we obtain $[f(z)/zf'(0)]^\gamma \ll (1 - z^m)^{-2\gamma/m}$, which is equivalent to the desired result.

THEOREM 1. *$\phi(z) \in B_m(\alpha)$ if and only if*

$$\phi(z) = [f(z^m)]^{1/m} \tag{2}$$

where $f(z) \in B_1(\alpha/m)$.

PROOF. (i) If $\phi(z) \in B_m(\alpha)$, then on differentiating (1) we obtain

$$z\phi'(z)\phi(z)^{\alpha-1} = g(z)^\alpha P(z), \tag{3}$$

where g is starlike and Re $P(z) > 0$. By Lemma 1 (a) there exists an $f \in S$ satisfying (2), and substitution of (2) in (3) yields

$$z^m f'(z^m) f(z^m)^{\alpha/m-1} = g(z)^\alpha P(z).$$

Replacing z by $e^{2k\pi i/m}z$ we have

$$z^m f'(z^m) f(z^m)^{\alpha/m-1} = g(e^{2k\pi i/m}z)^\alpha P(e^{2k\pi i/m}z), \tag{4}$$

for $k=0, 1, \cdots, m-1$. If we multiply the m equations in (4) and then take the mth root we obtain

$$z^m f'(z^m) f(z^m)^{\alpha/m-1} = \left[\prod_{k=0}^{m-1} g(e^{2k\pi i/m}z)\right]^{\alpha/m} \left[\prod_{k=0}^{m-1} P(e^{2k\pi i/m}z)\right]^{1/m}.$$

It is easily verified that the function $[\prod_{k=0}^{m-1} g(e^{2k\pi i/m}z)]^{1/m}$ is m-fold symmetric and starlike and hence, by Lemma 1 (b), can be written as $[h(z^m)]^{1/m}$ where h is starlike. It is also easily verified that

$$\left[\prod_{k=0}^{m-1} P(e^{2k\pi i/m}z)\right]^{1/m} = c_0 + c_m z^m + \cdots = Q(z^m),$$

say, where Re $Q(z) > 0$. Thus we have

$$z^m f'(z^m) f(z^m)^{\alpha/m-1} = h(z^m)^{\alpha/m} Q(z^m), \qquad zf'(z)f(z)^{\alpha/m-1} = h(z)^{\alpha/m}Q(z),$$

or $f \in B(\alpha/m)$.

(ii) Conversely, if $f \in B(\alpha/m)$ we have

$$zf'(z)f(z)^{\alpha/m-1} = h(z)^{\alpha/m}Q(z),$$

where h is starlike and Re $Q(z) > 0$. Thus

$$z^m f'(z^m) f(z^m)^{\alpha/m-1} = h(z^m)^{\alpha/m} Q(z^m)$$

and, if we let $\phi(z) = [f(z^m)]^{1/m}$, we obtain

$$z\phi'(z)\phi(z)^{\alpha-1} = [h(z^m)^{1/m}]^\alpha Q(z^m). \tag{5}$$

By Lemma 1 (b), $g(z)=[h(z^m)]^{1/m}$ is (m-fold symmetric) starlike, and if we write $P(z)=Q(z^m)$ we have

$$z\phi'(z)\phi(z)^{\alpha-1} = g(z)^\alpha P(z).$$

Thus $\phi(z)\in B_m(\alpha)$, and this completes the proof of the theorem.

THEOREM 2. *If* $\phi(z)\in B_m(\alpha)$ *then* $[\phi(z)/z]^\alpha \ll (1-z^m)^{-2\alpha/m}$.

PROOF. The function

$$F(z) = [\phi(z)/z]^\alpha = 1 + A_1 z^m + A_2 z^{2m} + \cdots$$

satisfies the differential equation

$$zF'(z) + \alpha F(z) = \alpha\phi'(z)\phi(z)^{\alpha-1}/z^{\alpha-1},$$

or, using (5),

$$zF'(z) + \alpha F(z) = \alpha\{[h(z^m)]^{1/m}/z\}^\alpha Q(z^m), \tag{6}$$

where $[h(z^m)]^{1/m}$ is m-fold symmetric and starlike and $\operatorname{Re} Q(z^m)>0$. Hence, by Lemma 2, we have

$$\{[h(z^m)]^{1/m}/z\}^\alpha \ll |h'(0)|^{\alpha/m}(1-z^m)^{-2\alpha/m}. \tag{7}$$

Also, if $Q(z^m)=c_0+c_m z^m+c_{2m}z^{2m}+\cdots$ then

$$Q(z^m) \ll |c_0| \frac{1+z^m}{1-z^m} \tag{8}$$

[4, p. 170]. Since multiplication preserves the majorization property, (6), (7) and (8) yield

$$zF'(z) + \alpha F(z) \ll \alpha |h'(0)|^{\alpha/m} |c_0| \frac{1+z^m}{(1-z^m)^{1+2\alpha/m}}\cdot$$

Setting $z=0$ in (6) we obtain $\alpha=\alpha[h'(0)]^{\alpha/m}c_0$, and consequently

$$zF'(z) + \alpha F(z) \ll \alpha \frac{1+z^m}{(1-z^m)^{1+2\alpha/m}}\cdot$$

On comparing coefficients we have

$$|(mn+\alpha)A_n| \leqq \alpha\left[\binom{2\alpha/m+n}{n}+\binom{2\alpha/m+n-1}{n-1}\right],$$

$$|A_n| \leqq \binom{2\alpha/m+n-1}{n},$$

or equivalently, $[\phi(z)/z]^\alpha \ll (1-z^m)^{-2\alpha/m}$.

COROLLARY. *If $\phi \in B_m(1/N)$, where N is a positive integer and $\phi(z) = z + a_{m+1}z^{m+1} + \cdots$ then*

$$|a_{mn+1}| \leqq \binom{2/m + n - 1}{n}. \tag{9}$$

In particular for $\phi \in B(1/N)$ we have $|a_n| \leqq n$ and for $\phi \in B_2(1/N)$ we have $|a_{2n+1}| \leqq 1$.

PROOF. With $\alpha = 1/N$ we have

$$[\phi(z)/z]^{1/N} \ll (1 - z^m)^{-2/mN}.$$

Multiplying this result by itself N times we obtain $\phi(z)/z \ll (1 - z^m)^{-2/m}$, which is equivalent to the desired result.

The inequality (9) is sharp, as can be seen by considering the function $z(1 - z^m)^{-2/m}$ which is in $B_m(1/N)$.

We conclude by constructing an example of a function $f \in B(1/2)$ such that f is not close-to-convex. With an appropriate $c > 0$, let $w = \phi(z) = z + \cdots$ be the odd close-to-convex function that maps Δ onto the w-plane slit along the half-lines Re $w \geqq 0$, Im $w = c$ and Re $w \leqq 0$, Im $w = -c$. Since $\phi \in B_2(1)$, by Theorem 1 we have $\phi(z) = [f(z^2)]^{1/2}$, where $f(z) \in B(1/2)$. But the transformation $\zeta = \xi + i\eta = f(z)$ maps Δ onto the ζ-plane slit along the portion of the parabola $\xi = (\eta/2c)^2 - c^2$ defined for $\eta \geqq 0$, and this slit clearly cannot be expressed as a union of half-lines. It follows by a well-known geometric criterion (see, for example, Bielecki and Lewandowski [**2**, p. 61]) that the domain is not close-to-convex.

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UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506

STATE UNIVERSITY OF NEW YORK, BROCKPORT, NEW YORK 14420