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A NOTE ON FINITELY ADDITIVE SET FUNCTIONS

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ABSTRACT. In this note, a counterexample to a conjecture of Yosida and Hewitt on finitely additive set functions is given.

Let A be a σ-field of subsets of a set X and let N be a proper subfamily of A which is closed under countable unions having the additional property that $N \in N$, $A \in A$, $A \subset N$ implies $A \in N$. Let M denote the Banach space (under total variation norm) of all bounded, real valued finitely additive set functions on (X, A) such that for every $N \in N$ and every $m \in M$, $m(N) = 0$. Let $M^* = \{m \in M : m(A) = 0 \text{ or } 1 \text{ for all } A \in A\}$. For each $A \in A$ if $U_A = \{m \in M^* : m(A) = 1\}$ then M^*, equipped with the topology for which $\{U_A : A \in A\}$ is a base, is a compact Hausdorff space and M is isometrically isomorphic to $C^*(M^*)$, the Banach space of all bounded real valued regular Borel measures on M^*.

Yosida and Hewitt [**1**] gave a necessary and sufficient condition for a member of M to be countably additive, in terms of its isomorphic image in $C^*(M^*)$, and conjectured that if M does not have any countably additive member at all, then every member $\bar{m}$ of $C^*(M^*)$ is confined to a closed nowhere dense G_δ in M^*. Lloyd [**2**] characterized purely finitely additive members of M in terms of their isomorphic images in $C^*(M^*)$ from which it follows that if M does not have any countably additive member at all then every member of $C^*(M^*)$ is confined to a countable union of closed nowhere dense G_δ sets in M^*, but offered no counterexample for the conjecture of Yosida and Hewitt. We give one below.

Let X be the closed unit interval $[0, 1]$, A Borel subsets of X and N first category Borel subsets of X. Since every countably additive Borel measure on X lives in a first category Borel set, no member of M is countably additive. But M^* in this case is separable (see [**3**, p. 94]). Let $\{m_1, m_2, \cdots\}$ be a countable dense subset of M^*. If we define for each Borel subset B of M^*, $\bar{m}(B) = \sum_{k=1}^{\infty} 2^{-k} I_B(m_k)$ where $I_B(\cdot)$ is the characteristic function of B, $\bar{m}$ is a countably additive Borel measure on M^*. Further $\bar{m}(B) = \sup_k \bar{m}(B \cap \{m_1, \cdots, m_k\})$ shows that $\bar{m}$ is regular, finite sets being compact

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in M^*. Clearly $\bar{m}$ cannot be confined to any closed nowhere dense subset of M^* and so the conjecture is false.

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