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OSCILLATION THEOREMS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH FUNCTIONAL ARGUMENTS

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ABSTRACT. The oscillatory behavior of $Y''(t) + P(t)Y(g(t)) = 0$ where $g(t) \to \infty$ as $t \to \infty$ is investigated. Sufficient conditions for the oscillation of $Y'(t)$ and $Y(t)$ are developed.

1. **Introduction.** The purpose of this paper is to study the oscillatory behavior of the differential equation

$$Y''(t) + P(t)Y(g(t)) = 0. \tag{1.1}$$

It is assumed throughout that $P(t)$ and $g(t)$ are continuous on $[a, \infty)$ and that $g(t) \to \infty$ as $t \to \infty$. A nontrivial solution to (1.1) is called oscillatory if it exists on a half-line and has arbitrarily large zeros. A solution is called nonoscillatory if it is eventually of one sign. Equation (1.1) is called oscillatory if every solution is oscillatory.

2. **Main results.** Waltman [**5**] and Bradley [**2**] have recently demonstrated that all solutions of (1.1) are oscillatory if $P(t) \geqq 0$ and $\int^\infty P(t)\,dt = \infty$.

If $g(t) \equiv t$, equation (1.1) reduces to the ordinary differential equation

$$Y''(t) + P(t)Y(t) = 0. \tag{2.1}$$

It is a well-known theorem of Wintner [**6**] and Leighton [**3**] that $\int^\infty P(t)\,dt = \infty$ is sufficient for equation (2.1) to be oscillatory even when $P(t)$ is not assumed positive. It is also known [**1**] that all solutions of the nonlinear equation

$$Y''(t) + P(t)h(Y(t)) = 0,$$
$$h(x)x > 0 \quad \text{for } x \neq 0,$$
$$h'(x) \geqq 0 \quad \text{for all } x,$$

oscillates if $\int^\infty P(t) = \infty$, where $P(t)$ is not assumed positive. On the basis

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of these two results, one might conjecture that the Leighton-Wintner oscillation theorem could be extended to equations of the form (1.1). The equation $Y''(t) + ((\sin t)/(2 - \sin t))Y(t - \pi) = 0$ demonstrates that this is not the case. This equation has the nonoscillatory solution $Y(t) = 2 + \sin t$ even though $\int^\infty ((\sin t)/(2 - \sin t))\,dt = \infty$. However this solution does have the property that $Y'(t)$ has arbitrarily large zeros. Theorem 2.1 below shows that the derivative of any solution of an equation of form (1.1) has arbitrarily large zeros if we assume $g'(t) \geqq 0$ and $\int^\infty P(t)\,dt = \infty$.

THEOREM 2.1. *Assume $g(t)$ is differentiable, $g'(t) \geqq 0$, and $\int^\infty P(t)\,dt = \infty$. Then $Y'(t)$ has arbitrarily large zeros for any solution $Y(t)$ of* (1.1).

PROOF. If $Y(t)$ oscillates then the conclusion of the theorem is certainly true. If $Y(t)$ is ultimately positive, then so is $Y(g(t))$. Suppose $Y'(t) > 0$ for all large t. Then $W(t) = Y'(t)/Y(g(t))$ satisfies the equation

$$W'(t) = -P(t) - W(t)\frac{Y'(g(t))}{Y(g(t))}g'(t) \leqq -P(t).$$

Integrating we obtain

$$W(x) \leqq W(\alpha) - \int_\alpha^x P(t)\,dt.$$

Since $\int^\infty P(t)\,dt = \infty$ we have a contradiction. Suppose now that $Y'(t) < 0$ for all large t. It is not difficult to see that $\int^\infty P(t)\,dt = \infty$ implies that there exists T sufficiently large so that

$$\int_T^t P(s)\,ds \geqq 0 \quad \text{for } t \geqq T.$$

Hence, we have

$$\int_T^t P(s)Y(g(s))\,ds = Y(g(t))\int_T^t P(s)\,ds - \int_T^t Y'(g(s))g'(s)\int_T^s P(r)\,dr\,ds \geqq 0, \qquad t \geqq T.$$

Now integrating equation (1.1) we have $Y'(t) \leqq Y'(T) < 0$ which contradicts the fact that $Y(t)$ is positive for large t. This completes the proof.

Considering the equation

$$Y''(t) + \lambda P(t)Y(t) = 0, \tag{2.2}$$

we shall call $P(t)$ a strongly oscillatory coefficient if (2.2) is oscillatory for all positive λ. If $P(t) \geqq 0$, Nehari [**4**] shows that

$$\limsup_{x\to\infty} x\int_x^\infty P(t)\,dt = \infty$$

is a necessary and sufficient condition for $P(t)$ to be a strongly oscillatory coefficient.

In Theorem 2.2 below, we shall consider the equation

$$Y''(t) + P(t)f(Y(t), Y(g(t))) = 0, \tag{2.3}$$

where P, g, and f are assumed continuous, and $f(y, W)$ has the sign of y and W when they have the same sign. We will show that if $P(t)$ is a positive strongly oscillatory coefficient and if $g(t)$ increases "sufficiently fast," then any solution of (2.3) that exists on a ray (a, ∞) is oscillatory. In order to avoid the assumption that $g(t)$ is differentiable, we introduce a differentiable minorant $h(t)$. This will allow the theorem to be applied to delay equations of the form $Y''(t) + P(t)Y(t - \tau(t)) = 0, 0 \leqq \tau(t) \leqq M$, where $\tau(t)$ is not assumed differentiable.

THEOREM 2.2. *If*

(i) $h(t) \leqq g(t)$ *and* $0 < k \leqq h'(t) \leqq 1$,

(ii) *there exists* $M > 0$ *such that* $y \geqq M$ *implies*

$$\liminf_{|w|\to\infty} \left|\frac{f(y, w)}{w}\right| \geqq \epsilon > 0,$$

(iii) $P(t) \geqq 0$ *and* $\limsup_{x\to\infty} x \int_x^\infty P(t)\, dt = \infty$,

then all solutions of (2.3) *existing on* (a, ∞) *are oscillatory.*

PROOF. Assume the contrary. Then (2.3) has a nonoscillatory solution $Y(t)$. Assume that $Y(t) > 0$ for large t (the case $Y(t) < 0$ can be treated similarly). It is easy to verify that $Y'(t) > 0$ for large t. Let $W(t) = Y'(t)/Y(h(t))$. Then $W(t)$ satisfies

$$W'(t) = -P(t)\frac{f(Y(t), Y(g(t)))}{Y(h(t))} - \frac{Y'(h(t))}{Y(h(t))} h'(t)W(t).$$

Since $Y'(t) > 0$ for large t, $\lim_{t\to\infty} Y(t)$ exists either as a finite or infinite limit. If $\lim_{t\to\infty} Y(t) = b$ is finite, then

$$\lim_{t\to\infty}\frac{f(Y(t), Y(g(t)))}{Y(g(t))} = \frac{f(b, b)}{b} > 0.$$

If $\lim_{t\to\infty} Y(t) = \infty$, then by (ii) we have that

$$\frac{f(Y(t), Y(g(t)))}{Y(g(t))} \geqq \epsilon > 0 \tag{2.4}$$

for large t. In either case, we have that (2.4) holds for sufficiently large t.

Now since $Y(t)$ is increasing for large t we have that

$$P(t)\frac{f(Y(t),\, Y(g(t)))}{Y(h(t))} \geqq P(t)\frac{f(Y(t),\, Y(g(t)))}{Y(g(t))} \geqq \epsilon P(t)$$

and also that

$$\frac{Y'(h(t))}{Y(h(t))}\, h'(t)W(t) \geqq kW^2(t).$$

Therefore $W'(t) \leqq -\epsilon P(t) - kW^2(t)$ and if we define $R(t) = kW(t)$, we obtain $R'(t) + R^2(t) + k\epsilon P(t) \leqq 0$. However, by a well-known theorem of Wintner [7] this implies that the equation $Y''(t) + k\epsilon P(t)Y(t) = 0$ is nonoscillatory. But this contradicts the fact that (iii) implies $P(t)$ is a strongly oscillatory coefficient.

In Theorem 2.2 we made an assumption on the rate of increase of $g(t)$. The equation

$$Y''(t) + \tfrac{1}{4}(t^3 \ln t)^{-1/2}\, Y(\ln t) = 0 \quad \text{for } 1 < t$$

demonstrates that the assumption is necessary. This equation has the nonoscillatory solution $Y(t) = t^{1/2}$ even though $P(t) = \frac{1}{4}(t^3 \ln t)^{-1/2}$ is a strongly oscillatory coefficient.

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