THE GAME QUANTIFIER[1]

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ABSTRACT. For structures that satisfy certain mild definability conditions we show the following result: *A relation* $R(x_1, \cdots, x_n)$ *has a positive first order inductive definition iff for some first order* $Q(y, x_1, \cdots, x_n)$,

$$R(x_1, \cdots, x_n)$$
$$\Leftrightarrow (\forall y_1 \exists y_2 \forall y_3 \exists y_4 \cdots)(\exists k)Q(\langle y_1, \cdots, y_k \rangle, x_1, \cdots, x_n).$$

This yields easily some of the basic results about inductive definability, e.g., the existence of a universal inductively definable set.

The theory of hyperarithmetical relations on the integers has been generalized to arbitrary first order structures in [5], [6]. Most of the known characterizations of hyperarithmeticalness hold in the abstract; however the basic Suslin-Kleene theorem holds only for *countable* structures (Barwise-Gandy-Moschovakis [1], Moschovakis [7]) while in general the *hyperprojective* subsets of a structure are a proper subclass of the Δ_1^1 sets. Because of this there has been no simple *syntactical* characterization of the hyperprojective sets. We give here such a characterization using infinite strings of alternating quantifiers in the manner of Keisler [3].

According to the normal form theorem (Kleene [4]), each Π_1^1 relation on ω is obtained by a single application of the *Suslin quantifier* (the dual of the classical operation $\mathcal{A}$) to an arithmetical (or even recursive) relation:

$$(1) \quad \begin{aligned} R(x_1, \cdots, x_n) &\Leftrightarrow (\mathcal{S}y)Q(y, x_1, \cdots, x_n) \\ &\Leftrightarrow (\forall y_1 \forall y_2 \forall y_3 \forall y_4 \cdots)(\exists k)Q(\langle y_1, \cdots, y_k \rangle, x_1, \cdots, x_n). \end{aligned}$$

It is rather amusing and easy to prove that we get another normal form for Π_1^1 relations if we use the *game quantifier* instead, i.e. if we change every other $\forall$ to $\exists$:

$$(2) \quad \begin{aligned} R(x_1, \cdots, x_n) &\Leftrightarrow (\mathcal{G}y)Q(y, x_1, \cdots, x_n) \\ &\Leftrightarrow (\forall y_1 \exists y_2 \forall y_3 \exists y_4 \cdots)(\exists k)Q(\langle y_1, \cdots, y_k \rangle, x_1, \cdots, x_n). \end{aligned}$$

(Of course we must give a correct interpretation to the infinite alternating string in (2); see §1.) Now (1) and (2) make sense not only on the integers but over any structure in which we can define codings for tuples, in particular, any *acceptable* structure $\mathfrak{A}$ in the sense of Moschovakis [7]. For some of these structures the Suslin quantifier preserves first order definability, because we can code infinite sequences by single elements of the structure. This is in fact true for the standard model of second order number theory with domain $\omega \cup {}^{\omega}\omega$ which provided the initial motivation for studying hyperprojective relations. The game quantifier however goes beyond first order definability and gives a normal form for the abstract analog of $\mathbf{\Pi}_1^1$.

DEFINITION. Let $\mathfrak{A} = \langle A, R_1, \cdots, R_l, f_1, \cdots, f_m \rangle$ be acceptable. A relation $R(x_1, \cdots, x_n)$ on A is $\mathbf{G}_1$ if for some (first order) definable (with parameters from A) relation $Q(y, x_1, \cdots, x_n)$,

$$(3) \qquad\qquad R(x_1, \cdots, x_n) \Leftrightarrow (\mathbf{G}y)Q(y, x_1, \cdots, x_n).$$

MAIN RESULT. *Let $\mathfrak{A}$ be acceptable. The $\mathbf{G}_1$ relations on A are precisely the inductively definable (or semihyperprojective) relations on A.*

One can easily piece together a proof of this result using the techniques developed in [5] and in particular the game-theoretic characterization of semihyperprojectivity given in §19 of that paper. The point of this note is that we give a direct and elementary proof of the normal form (3) for *inductively definable relations*, a proof that does not depend on the complicated calculus of hyperprojective functions developed in [5]. In §3 we indicate how this can be used to give fairly direct proofs of the basic theorems about inductive definability, e.g. the existence of a universal inductively definable set.

To save space, we shall assume the notation and terminology established in §1–§3 of [7]. The result of that paper however is not essential for understanding the main argument in this note.

1. Elementary transformations. The operator $\mathbf{S}$ on an acceptable structure can be immediately understood as quantification over all countable sequences in A:

$$(\mathbf{S}y)Q(y) \Leftrightarrow (\forall f \in {}^{\omega}A)(\exists k)Q(\langle f(1), \cdots, f(k) \rangle).$$

To interpret $(\mathbf{S}y)Q(y)$, consider the infinite two-player game where the first player (call him $\forall$) chooses y_1, the second player (call him $\exists$) chooses y_2, then $\forall$ chooses y_3, then $\exists$ chooses y_4, etc. ad infinitum, and $\exists$ wins iff for some k, $Q(\langle y_1, \cdots, y_k \rangle)$. Put

$$(4) \qquad\qquad (\mathbf{S}y)Q(y) \Leftrightarrow \exists \text{ has a winning strategy.}$$

Strategies are functions on finite tuples from A to A. If $\sigma * \tau$ is the play

resulting when $\forall$'s strategy σ is pitted against $\exists$'s strategy τ, we can rewrite (4) as

$$(5) \qquad (\mathcal{G}y)Q(y) \Leftrightarrow (\exists\tau)(\forall\sigma)(\exists k)Q(\langle \sigma * \tau(1), \cdots, \sigma * \tau(k)\rangle).$$

This game is *open* in the terminology of Gale-Stewart [2] and by the basic result of that paper it is *determined*, i.e. either $\forall$ or $\exists$ must have a winning strategy. Thus we can also write

$$(6) \qquad (\mathcal{G}y)Q(y) \Leftrightarrow (\forall\sigma)(\exists\tau)(\exists k)Q(\langle \sigma * \tau(1), \cdots, \sigma * \tau(k)\rangle).$$

Another easy consequence of the Gale-Stewart theorem is that

$$\neg\, (\mathcal{G}y)Q(y) \Leftrightarrow (\mathcal{G}^{\smile} y)\, \neg\, Q(y),$$

where $\mathcal{G}^{\smile}$ is the dual quantifier defined by

$$(\mathcal{G}^{\smile} y)Q(y) \Leftrightarrow (\exists y_1 \forall y_2 \exists y_3 \forall y_4 \cdots)(\forall k)Q(\langle y_1, \cdots, y_k\rangle).$$

It is easy to verify the following equivalences. (Recall from [7] that if Seq (x), then $K(x)$ is the length of the sequence that x codes and $(x)_i$ $(1 \leqq i \leqq K(x))$ is the ith term of that sequence.)

$$(7) \qquad (\mathcal{G}x)P(x) \ \& \ (\mathcal{G}y)Q(y) \Leftrightarrow (\mathcal{G}x)(\mathcal{G}y)[P(x) \ \& \ Q(y)].$$

$$(8) \qquad (\mathcal{G}x)P(x) \lor (\mathcal{G}y)Q(y) \Leftrightarrow (\mathcal{G}x)(\mathcal{G}y)[P(x) \lor Q(y)].$$

$$(9) \qquad (\forall x)P(x) \Leftrightarrow (\mathcal{G}x)[\text{Seq } (x) \ \& \ K(x) \geqq 1 \ \& \ P((x)_1)].$$

$$(10) \qquad (\exists x)P(x) \Leftrightarrow (\mathcal{G}x)[\text{Seq } (x) \ \& \ K(x) \geqq 2 \ \& \ P((x)_2)].$$

$$(11) \qquad (\mathcal{S}x)P(x) \Leftrightarrow (\mathcal{G}x)Q(x),$$

where

$$(12) \qquad Q(x) \Leftrightarrow \text{Seq } (x) \ \& \ K(x) \text{ is odd } \& \ P(\langle(x)_1, (x)_3, \cdots, (x)_{K(x)}\rangle).$$
$$(\mathcal{G}x)(\mathcal{G}y)P(x, y) \Leftrightarrow (\mathcal{G}z)Q(z),$$

where

$$Q(z) \Leftrightarrow \{\text{Seq } (z) \ \& \ (\exists m < K(z))[m \text{ is even } \& \ P(\langle(z)_1, \cdots, (z)_m\rangle,$$
$$\langle(z)_{m+1}, \cdots, (z)_{K(z)}\rangle)]\}$$
$$\lor \{\text{Seq } (z) \ \& \ (\exists m < K(z))[m \text{ is odd } \& \ P(\langle(z)_1, \cdots, (z)_m\rangle,$$
$$\langle(z)_{m+2}, \cdots, (z)_{K(z)}\rangle)]\}.$$

From these it follows immediately that *the class of* $\mathcal{G}_1$ *relations on an acceptable structure contains all definable relations and is closed under the operations* $\&, \lor, \forall, \exists, \mathcal{S}, \mathcal{G}$.

2. The main argument. Following §3 of [7], a relation $R(x_1, \cdots, x_n)$ on A is inductively definable iff for some definable $f(x_1, \cdots, x_n)$,

$$(13) \qquad R(x_1, \cdots, x_n) \Leftrightarrow f(x_1, \cdots, x_n) \in \bigcup_\xi P^\xi$$

where the sets P^ξ are defined inductively $x \in P^\xi \Leftrightarrow P(x, \bigcup_{\eta < \xi} P^\eta)$ relative to a given first order formula $P(x, S)$ with only positive occurrences of S.

If $R(x_1, \cdots, x_n)$ is $\mathfrak{S}_1$ satisfying (3) with a definable Q, it is easy to verify that R satisfies (13) with

$$P(x, S) \Leftrightarrow (\exists z)(\exists w)[x = \langle z, w \rangle \ \& \ \text{Seq } (w) \ \& \ K(w) = n \ \& \ \text{Seq } (z)$$
$$\& \ \{Q(z, (w)_1, \cdots, (w)_n) \lor [K(z) \text{ is even} \ \& \ (\forall t)(z * \langle t \rangle \in S)]$$
$$\lor \ [K(z) \text{ is odd} \ \& \ (\exists t)(z * \langle t \rangle \in S)]\}],$$
$$f(x_1, \cdots, x_n) = \langle 1, \langle x_1, \cdots, x_n \rangle \rangle.$$

(Recall that 1 codes the empty sequence.)

To prove the converse, we first notice that each first order $P(x, S)$ with only positive occurrences of S can be put in the form

$$(14) \qquad P(x, S) \Leftrightarrow (\mathbf{Q}z)(\forall y)[D(x, z, y) \Rightarrow y \in S],$$

where $D(x, z, y)$ is first order and $\mathbf{Q}$ is a finite string of quantifiers. (This can be shown easily by putting $P(x, S)$ in prenex normal form and then putting the matrix in conjunctive normal form.) Now let $P^\infty = \bigcup_\xi P^\xi$ and notice that

$$x \notin P^\infty \Leftrightarrow (\mathbf{Q}^\vee z)(\exists y)[D(x, z, y) \ \& \ y \notin P^\infty],$$

where $\mathbf{Q}^\vee$ is the string dual to $\mathbf{Q}$; if we apply the same equivalence to the clause $y \notin P^\infty$ within the brackets, we get

$$x \notin P^\infty \Leftrightarrow (\mathbf{Q}^\vee z_1)(\exists y_1)(\mathbf{Q}_2^\vee z_2)(\exists y_2)[D(x, z_1, y_1) \ \& \ D(y_1, z_2, y_2) \ \& \ y_2 \notin P].$$

A few more expansions of this form suggest the extrapolation to the infinite

$$(15) \quad \begin{array}{l} x \notin P^\infty \\ \Leftrightarrow (\mathbf{Q}^\vee z_1)(\exists y_1)(\mathbf{Q}^\vee z_2)(\exists y_2) \cdots [D(x, z, y_1) \ \& \ D(y_1, z_2, y_2) \ \& \cdots], \end{array}$$

where P^∞ does not occur on the right anymore. A rigorous proof of direction $\Rightarrow$ of (15) is easy. To give a rigorous proof of direction $\Leftarrow$, assume that the right-hand side holds for some $x \in P^\infty$ and work for a contradiction. For some $\xi = \xi(x)$ we have $x \in P^\xi$ so that

$$(16) \qquad (\mathbf{Q}z)(\forall y)\left[D(x, z, y) \Rightarrow y \in \bigcup_{\eta < \xi} P^\eta \right];$$

this means that in the $\forall$-$\exists$ finite game that determines the truth of (16), $\exists$ has a winning strategy, call it σ_x. *Now let player $\forall$ follow this strategy σ_x*

against ∃'s *winning strategy in the game determined by the right-hand side of* (15). After a finite number of moves we obtain a finite sequence z_1, y_1 such that on the one hand $D(x, z_1, y_1)$ and on the other $D(x, z_1, y_1) \Rightarrow y_1 \in \bigcup_{\eta < \xi(x)} P^\eta$, so that for some $\eta = \xi(y_1) < \xi(x)$ we have $y_1 \in P^\eta$. We now repeat the argument with y_1 in the place of x and so on ad infinitum, so that we get an infinite descending chain of ordinals $\xi(x) > \xi(y_1) > \xi(y_2) > \cdots$, which is the desired contradiction.

Taking the negation on the left and the dual on the right of (15), we have

$$(17) \quad x \in P^\infty \Leftrightarrow (Qz_1 \forall y_1 Qz_2 \forall y_2 \cdots)(\exists k)[D(x, z_1, y_1) \vee D(y_{k-1}, z_k, y_k)];$$

since Q could have been chosen to be an alternating string beginning and ending with ∃, (17) implies almost immediately that P^∞ and hence R is $\mathbf{G}_1$.

3. **Consequences.** A set X is *universal* for a class Γ of subsets of A if $X \in \Gamma$ and each set in Γ is of the form $\{x : \langle a, x \rangle \in X\}$ for some $a \in A$. Suppose there is a class $\mathfrak{F}$ of definable relations on A which contains the relations defined by quantifier free formulas, which contains the relations $x \in N$, Seq (x), $m \leq n$, which is closed under &, $\vee$, $(\exists m < n)$ and substitution of the functions $\langle x_1, \cdots, x_k \rangle$, $(x)_i$ and such that the class of sets (unary relations) in $\mathfrak{F}$ has a universal set. The transformations in §1 then show easily that the class of $\mathbf{G}_1$ sets has a universal set. In many cases such classes $\mathfrak{F}$ are well known, e.g., for the standard model of second order number theory we can take $\mathfrak{F} = \Sigma_1^0$. That there is such a $\mathfrak{F}$ for each acceptable structure follows easily from some of the elementary results about prime computability in [5]. Thus *for each acceptable $\mathfrak{A}$ there is a universal $\mathbf{G}_1$ set, hence a universal inductively definable set.* One can now assign ordinals to the elements of this set and then prove a *prewellordering theorem* like Theorem 7 of [5] using directly an inductive definition (cf. the proof of Lemma 2.6 in [1]). This is sufficient to yield by standard methods most of the basic results about inductively definable relations, including reduction for $\mathbf{G}_1$, a hierarchy on $\mathbf{G}_1 \cap \mathbf{G}_1^\vee$, etc.

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