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2-GENERATOR GROUPS AND PARABOLIC CLASS NUMBERS

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ABSTRACT. It is shown that if x, y are generators of the finite group G such that $x^p = y^q = (xy)^n = 1$, where p, q, n are integers > 1, $(p, q) = 1$, and xy is of true order n, then the order $\mu = nt$ of G satisfies $n \leqq pqt^p$. This result is used to show that if F is a Fuchsian group of genus 0 generated by 2 elliptic elements of coprime order and with 1 parabolic class, then F possesses only finitely many normal subgroups having a given number of parabolic classes.

Introduction. Let $\Gamma = LF(2, Z)$ be the classical modular group, so that Γ is the free product of the cyclic group $\{T\}$ of order 2 and the cyclic group $\{U\}$ of order 3, where $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, and $S = TU = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is parabolic. Let G be a normal subgroup of Γ of index μ and level n (the least positive integer such that $S^n \in G$). Then if t is the number of parabolic classes of G, $t = \mu/n$. In [3] the conjecture was made that there are only finitely many normal subgroups of Γ having a given number t of parabolic classes. This conjecture was proved by L. Greenberg in his paper [2], who showed that $n \leqq 6t^3$, so that $\mu \leqq 6t^4$.

In connection with his work on automorphisms of closed Riemann surfaces, R. Accola [1] proved a number of group-theoretic lemmas about finite groups generated by 2 elements one of which is of period 2. A consequence of his work is the improvement of the inequality above to $n \leqq 6t^2$ always, and $n \leqq t^2$ if Γ/G is not abelian.

In this paper we apply Accola's method with some modifications to the more general situation of any finite group G generated by 2 elements, whose orders are assumed relatively prime for our purpose. We obtain a number of inequalities connecting the order of G with the orders of the elements and their products. The results are used to show that Fuchsian groups of genus 0 with two elliptic generators of coprime order and with one parabolic class have the same sort of property as Γ with respect to the parabolic class number.

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The group-theoretical results. We have the following lemma:

LEMMA 1. *Suppose that the group G possesses a central normal subgroup C of index n. Let m be any multiple of n. Then the mapping $g \to g^m$, $g \in G$, is a homomorphism.*

PROOF. The mapping $g \to g^n$ is just the transfer homomorphism of G into C, so the result follows (see [**4**, Chapter V]).

Now let p, q, n be integers > 1 such that $(p,q) = 1$. Let $G = \{x, y\}$ be a group of order μ such that $x^p = y^q = (xy)^n = 1$, and $z = xy$ is of true order n. Since $n \mid \mu$, we may write $\mu = nt$.

We are interested in inequalities involving μ, n, t. We shall prove

THEOREM 1. *The quantities μ, n, t satisfy the following inequalities:*

$$n \leqq pqt^p, \tag{1}$$

$$\mu \leqq pqt^{p+1}, \tag{2}$$

$$\mu \geqq (pq)^{-1/p} n^{1+1/p}. \tag{3}$$

PROOF. The inequalities are entirely equivalent, and it is sufficient to prove any one of them. Put $C = \bigcap_{i=0}^{p-1} \{x^i z x^{-i}\}$. Then clearly, C is normalized by z (since C is a subgroup of the cyclic group $\{z\}$), and also, C is normalized by x. Thus C is in fact a normal subgroup of G, since x and z generate G. Also for some divisor k of n, $C = \{z^k\}$, so that C is of order n/k and index kt. Furthermore, Poincaré's index inequality implies that

$$(G:C) \leqq \prod_{i=0}^{p-1} (G:\{x^i z x^{-i}\}), \qquad kt \leqq t^p, \qquad k \leqq t^{p-1}.$$

We now remark that the elements $x^i z x^{-i}$, $0 \leqq i \leqq p-1$, generate G. For if

$$H = \{z, xzx^{-1}, \cdots, x^{p-1} z x^{-(p-1)}\},$$

then z normalizes H (since $z \in H$) and x normalizes H, so that H is a normal subgroup of G. Since $xy \in H$, we have that, in G/H, $x^p = y^q = 1$, $xy = 1$, so that $x = y^{-1}$, $x^q = y^{-q} = 1$. Since $(p,q) = 1$, $x = 1$; and thus also $y = 1$. It follows that G/H is trivial, and so $G = H$.

Now $x^{-i} z^k x^i = z^{kl_i}$, $0 \leqq i \leqq p-1$, since C is a cyclic normal subgroup of G generated by z^k. Hence

$$\begin{aligned}(x^i z x^{-i})^{-1} z^k (x^i z x^{-i}) &= x^i z^{-1} x^{-i} z^k x^i z x^{-1} = x^i z^{-1} z^{kl_i} z x^{-1} \\ &= x^i z^{kl_i} x^{-1} = z^k.\end{aligned}$$

It follows that C is a *central* normal subgroup of G. By Lemma 1, the mapping $\varphi: g \to g^{pqkt}$, $g \in G$, is a homomorphism. Since $\varphi: x \to 1$ and $\varphi: y \to 1$, $\varphi: g \to 1$ for all $g \in G$. In particular, $z^{pqkt} = 1$. It follows that

$pqkt \equiv 0 \bmod n$, and hence $n \leqq pqkt \leqq pqt^p$, which is just (1). This completes the proof.

The restriction that $(p, q) = 1$ is essential, as may be seen from the dihedral group, for example.

An application. As a consequence of Theorem 1, we prove

THEOREM 2. *Let F be the Fuchsian group with presentation*

$$E_1^p = E_2^q = 1, \qquad E_1E_2P = 1,$$

where P is parabolic and p, q are relatively prime integers > 1. *Then F has only finitely many normal subgroups of finite index having a given number t of parabolic classes.*

PROOF. Let G be a normal subgroup of F of index μ having t parabolic classes. Then if P is of exponent n modulo G, $t = \mu/n$. In the group F/G, which is of order $\mu = nt$, we have

$$E_1^p = E_2^q = (E_1E_2)^n = 1,$$

and E_1E_2 is of true order n. By the previous theorem, we have $\mu \leqq pqt^{p+1}$. Since the index is bounded, and since F is finitely generated, there are only finitely many possibilities for G. This completes the proof.

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