PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 1, January 1972

GENERALIZED POSITIVE LINEAR FUNCTIONALS ON A BANACH ALGEBRA WITH AN INVOLUTION[1]

PARFENY P. SAWOROTNOW

ABSTRACT. Let A be a proper H^*-algebra and let B be a Banach $*$-algebra with an identity. A linear mapping $\varphi: B \to A$ is called a positive A-functional if $\sum_{i,j} a_i^* \varphi(x_i^* x_j) a_j$ is positive for all $x_1, x_2, \cdots, x_n \in B$ and $a_1, a_2, \cdots, a_n \in A$. It is shown that for each positive A-functional φ there exists a $*$-representation T of B by A-linear operators on a Hilbert module H such that $\varphi(x) = (f_0, Txf_0)$ for all $x \in B$ and some $f_0 \in H$. If B is of the form $B = \{\lambda e + x \mid \lambda \text{ complex}, e \text{ is the (abstract) identity}, x \in L^1(G)\}$ for some locally compact group G then φ has the form $\varphi(\lambda e + x) = \lambda\varphi(e) + \int_G x(t)p(t)\,dt$ for some generalized (A-valued) positive definite function p on G, $x \in L^1(G)$.

1. The present work is a continuation of the study of Hilbert modules [6], [7]. In the previous paper [7] we generalized the theorem which states that each positive definite function on a group G is of the form $p(t) = (U_t f_0, f_0)$ for some unitary representation U of G, where f_0 is some member of the Hilbert space on which U acts. In this paper we will generalize the concept of a positive linear functional on a $*$-algebra and will prove that each generalized positive functional φ on a Banach algebra B is of the form $\varphi(x) = (f_0, Txf_0)$ for some $*$-representation $x \to Tx$ of B by A-linear operators on some Hilbert module.

Applying this result to a group algebra we shall derive an integral representation of the generalized positive linear functional on $L^1(G)$ in terms of an H^*-algebra valued positive definite function on G. In this way we will establish generalizations of Theorem 2 in §17 and Theorem 2 in §30 of [5].

2. Let A be a proper H^*-algebra [1] and let $\tau(A) = \{xy \mid x, y \in A\}$ be its trace-class [8]. It was shown in [8] that $\tau(A)$ is a Banach algebra with respect to some norm $\tau(\ )$ which is related to the norm $|\ |$ of A by the identity "$\tau(a^*a) = |a|^2, a \in A$". There is a partial ordering defined on

---

Presented to the Society, April 9, 1971; received by the editors November 25, 1970.

AMS 1970 *subject classifications*. Primary 22D20, 46K15, 46H25; Secondary 46K10, 43A35.

Key words and phrases. Hilbert module, H^*-algebra, Banach $*$-algebra, group algebra, $*$-representation.

[1] This research was supported by NSF Grant GP-11118.

$\tau(A)$ by the requirement that $a \geqq 0$ if $a = b^*b$ for some $b \in A$. Also there is a trace tr defined on $\tau(A)$ such that $\operatorname{tr} a = \tau(a)$ if $a \geqq 0$ and $\operatorname{tr}(xy^*) = \operatorname{tr}(y^*x) = (x \,.\, y)$ for all $x, y \in A$ (here (.) denotes the scalar product on A). For further details on $\tau(A)$ the reader is referred to **[8]** and **[9]**.

A right module H over A is called a Hilbert A-module if there exists a $\tau(A)$-valued function (,) on $H \times H$ having the following properties:

(i) If $f, g, h \in H$ and $a \in A$ then $(f+g, h) = (f, h) + (g, h)$, $(f, g)^* = (g, f)$, $(f, ga) = (f, g)a$, $(f, f) \geqq 0$ and $|\operatorname{tr}(f, g)|^2 \leqq \tau(f, f)\tau(g, g)$.

(ii) $(f, f) = 0$ if and only if $f = 0$.

(iii) H is complete in the norm $\|f\| = (\tau(f, f))^{1/2}$.

The function (,) is called generalized scalar product. There is a linear structure on H such that H is an ordinary Hilbert space with respect to the scalar product $[f, g] = \operatorname{tr}(g, f)$. An A-linear operator on H is an additive mapping $T: H \to H$ such that $T(fa) = (Tf)a$ for all $f \in H$, $a \in A$; T is bounded in $\|Tf\| \leqq M \|f\|$ for some $M \geqq 0$ and all $f \in H$. Each bounded A-linear operator T is linear and its adjoint T^* has the property that $(Tf, g) = (f, T^*g)$ for all $f, g \in H$.

3. Let B be a Banach algebra with the identity e and an involution $x \to x^*$ such that $|x^*| = |x|$ for all $x \in B$ and let A be a proper H^*-algebra (| | denotes the norm for each algebra).

DEFINITION. A linear mapping $\varphi: B \to A$ is called a positive A-functional if $\sum_{i,j} a_i^* \varphi(x_i^* x_j) a_j \geqq 0$ for all $x_1, x_2, \cdots, x_n \in B$ and $a_1, a_2, \cdots, a_n \in A$.

LEMMA 1. *If φ is a positive A-functional on B then $\varphi(x^*) = \varphi(x)^*$ for all $x \in B$.*

PROOF. If $a \in A$ then the mapping $x \to \operatorname{tr}(a^*\varphi(x)a)$ is a positive linear functional on B and so $\operatorname{tr}(a^*\varphi(x^*)a) = (\operatorname{tr})^-(a^*\varphi(x)a)$ for all $x \in B$, as it was shown on p. 96 of **[3]**.[2]

Now let $a, b \in A$. Then

$$\begin{aligned}
4(\varphi(x^*)a \,.\, b) &= 4 \operatorname{tr} b^*\varphi(x^*)a \\
&= \operatorname{tr}(a+b)^*\varphi(x^*)(a+b) - \operatorname{tr}(a-b)^*\varphi(x^*)(a-b) \\
&\quad + i \operatorname{tr}(a+ib)^*\varphi(x^*)(a+ib) \\
&\quad - i \operatorname{tr}(a-ib)^*\varphi(x^*)(a-ib) \\
&= (\operatorname{tr})^-[(a+b)^*\varphi(x)(a+b)] - (\operatorname{tr})^-[(a-b)^*\varphi(x)(a-b)] \\
&\quad + i(\operatorname{tr})^-[(a+ib)^*\varphi(x)(a+ib) \\
&\quad - i(\operatorname{tr})^-[(a-ib)^*\varphi(x)(a-ib)] \\
&= 4(\operatorname{tr})^-(a^*\varphi(x)b) = 4 \operatorname{tr}(\varphi(x)b)^*a = 4(a \,.\, \varphi(x)b).
\end{aligned}$$

Thus $\varphi(x^*) = \varphi(x)^*$ since a, b were arbitrary.

[2] Here and below $(\operatorname{tr})^-(\cdots)$ denotes the complex-conjugate of $\operatorname{tr}(\cdots)$.

Let H be a Hilbert A-module, let $x \to Tx$ be a representation of B by bounded A-linear operators on H (T is a homomorphism such that $Te = I$ and $Tx^* = T^*x$ for all $x \in B$) and let $f_0 \in H$. Then $\varphi(x) = (f_0, Txf_0)$ is a positive A-functional on B:

$$\sum_{i,j} a_i^* \varphi(x_i^* x_j) a_j = \sum_{i,j} a_i^* (f_0, Tx_i^* Tx_j f_0) a_j = \left(\sum_i Tx_i f_0 a_i, \sum_i Tx_i f_0 a_i \right) \geqq 0.$$

The converse is also true as it is stated in Theorem 1. A *-representation T of B is said to be regular if $Tx(f) = 0$ for all $x \in B$ implies $f = 0$ (in the terminology of Naimark [**5**, §29]: T has no degenerate subrepresentations).

THEOREM 1. *For each positive A-functional φ on a Banach algebra B with an isometric involution (and an identity) there exists a Hilbert A-module H, a regular *-representation $x \to Tx$ by bounded A-linear operators on H and $f_0 \in H$ such that $\varphi(x) = (f_0, Txf_0)$ for all $x \in B$.*

PROOF. Let K be the set of all formal expressions $f = \sum_{i=1}^n x_i a_i$ with $x_i \in B$, $a_i \in A$; if $g = \sum y_i b_i$ with $y_i \in B$, $b_i \in A$ define $(f, g) = \sum_{i,j} a_i^* \varphi(x_i^* y_i) b_j$. Then it is easy to see that (,) has all the properties of a generalized scalar product, except that (f, f) may be zero without f being a zero expression. We define $\|f\| = (\tau(f, f))^{1/2} = (\operatorname{tr}(f, f))^{1/2}$. Then $\| \ \|$ is a seminorm on K and one can show as in Theorem 2 of [**6**] that $\tau(f, g) \leqq \|f\| \cdot \|g\|$ for all $f, g \in K$.

Let $\mathfrak{N} = \{f \in K \mid (f, f) = 0\}$; then the last inequality implies that $(f, g) = 0$ for all $f \in \mathfrak{N}$, $g \in K$. Also $\mathfrak{N}$ is an A-submodule of K (since $\tau(fa, fa) = \tau((f, f)aa^*) \leqq \tau(f, f) \cdot \tau(aa^*)$ if $a \in A$). Let $H' = K/\mathfrak{N}$ and let H be the completion of H' with respect to the norm of H' which is induced by $\| \ \|$ (we will denote this norm also by $\| \ \|$). Then H is a Hilbert module.

For each $x \in B$ we define an operator $T'x$ on K by setting $T'x(f) = T'x(\sum_i x_i a_i) = \sum_i x x_i a_i$; then $T'x$ is A-linear and we shall show that $\|T'x(f)\| \leqq |x| \cdot \|f\|$ for all $f \in K$. This inequality will imply both that $T'x$ induces some operator Tx on H' and that this operator Tx is bounded.

So let $f = \sum x_i a_i$ in K be fixed. The linear functional

$$\psi(y) = \operatorname{tr} \sum_{i,j} a_i^* \varphi(x_i^* y x_j) a_j$$

on B is positive and so it follows from the Section 4 in §10 of [**5**] that $|\psi(y)| \leqq |y| \, \psi(e)$ for all $y \in B$. Taking $y = x^*x$ we have:

$$\begin{aligned} \|T'x(f)\|^2 &= \operatorname{tr}(T'xf, T'xf) = \operatorname{tr} \sum_{i,j} a_i^* \varphi(x_i^* x^* x x_j) a_j = \psi(x^*x) \\ &\leqq |x^*x| \cdot \psi(e) \leqq |x^*| \cdot |x| \cdot \operatorname{tr} \sum_{i,j} a_i^* \varphi(x_i^* x_i) a_j = |x|^2 \cdot \|f\|^2. \end{aligned}$$

Thus $\|T'xf\| \leqq |x| \cdot \|f\|$ for all $f \in K$ and so $T'x$ induces a bounded

A-linear operator Tx on H. We define $f_0 = \lim_n ee^n$, where e is the identity of B and e^n is as in the proof of Theorem 1 of [7] ($p(1)$ should be replaced by $\varphi(e)$ in the condition "$p(1)e = ep(1) = 0 \cdots$").

4. If the algebra B has no identity then we can adjoin it to B and consider the algebra $B_e = \{\lambda e + x \mid x \in B, \lambda \text{ complex}\}$ as it was done, for example, on p. 25 of [4] (or on p. 59 of [3]). We extend the involution to B_e by setting $(\lambda e + x)^* = \bar{\lambda} e + x^*$ and consider the norm $\|\lambda e + x\| = |\lambda| + \|x\|$.

Definition. A positive A-functional on B is a mapping $\varphi: B \to A$ such that there exists a positive A-functional φ' on B_e whose restriction to B coincides with φ.

This definition enables us to apply Theorem 1 to a group algebra in order to obtain a generalization of Theorem 2 in §30 of [5], which establishes the correspondence between positive definite functions defined on a topological group and (extendable) positive linear functionals defined on the group algebra.

We will need a few lemmas.

Lemma 2. *Let H be a Hilbert module over a proper H^*-algebra A and let T be a right centralizer* [8] *on A. Then there exists a bounded linear operator T' on H such that $T(f, g) = (T'f, g)$ for all $f, g \in H$.*

Proof. For a fixed $f \in H$ the mapping $b: g \to T(f, g)$ is a bounded A-linear functional on H ($\tau(T(f, g)) \leqq \|T\| \cdot \tau(f, g) \leqq \|T\| \cdot \|f\| \cdot \|g\|$ [9]). Thus [6, Theorem 3] there exists $z_f \in H$ such that $T(f, g) = (z_f, g)$ for all $g \in H$ and $\|b\| = \|z_f\| \leqq \|T\| \cdot \|f\|$. We define $T'f = z_f \cdots$.

Now let (S, μ) be a measurable space and let $h(s)$ be a mapping of S into a Hilbert module H such that the $\tau(A)$-valued function $t(s) = (g, h(s))$ is Pettis integrable for each $g \in H$. This in turn means that the scalar valued function $\varphi_m(s) = m(g, h(s))$ is Lebesgue integrable for each $m \in \tau(A)^*$ and there exists a member $(P) \int (g, h(s))\, ds$ of $\tau(A)$ such that $m((P) \int (g, h(s))\, ds) = \int m(g, h(s))\, d\mu(s)$ for all $m \in \tau(A)^*$. This condition could be restated as follows [9]: for each right centralizer T on A the mapping $s \to \operatorname{tr} T(g, h(s))$ is Lebesgue integrable and there exists $(P) \int (g, h(s))\, ds \in \tau(A)$ such that

$$\operatorname{tr} T\left((P)\int (g, h(s))\, ds\right) = \int \operatorname{tr} T(g, h(s))\, d\mu(s) \quad \text{for all } T \in R(A).$$

Definition. We shall say that an H-valued function $h(s)$ is P-integrable if $(g, h(s))$ is Pettis integrable for all $g \in H$ and there exists $P(h) \in H$ such that $(g, P(h)) = (P) \int (g, h(s))\, ds$ for all $g \in H$.

It was shown in [6] that H has also a structure of a Hilbert space with respect to the (ordinary) scalar product $[f, g] = \operatorname{tr}(g, f)$. Therefore one may speak about the Pettis integral of an H-valued function. It turns out that both integrals coincide:

LEMMA 3. *A Hilbert module valued function $h(s)$ is P-integrable if and only if it is Pettis integrable; also $P(h) = (P)\int h(s)\,ds$.*

PROOF. Taking T to be the identity operator we see at once that each P-integrable function is Pettis integrable.

Conversely let $h(s)$ be Pettis integrable. Then for each $g \in H$ and $T \in R(A) = \tau(A)^*$ [9] the function $\xi(s) = \operatorname{tr} T(g, h(s)) = \operatorname{tr}(T'_g, h(s))$ is Lebesgue integrable and

$$\int \operatorname{tr} T(g, h(s))\,d\mu(s) = \int \operatorname{tr}(T'_g, h(s))\,d\mu(s) = \operatorname{tr}\left(T'_g, (P)\int h(s)\,ds\right)$$
$$= \operatorname{tr} T\left(g, (P)\int h(s)\,ds\right),$$

which means that $(g, (P)\int h(s)\,ds)$ is the Pettis integral of $(g, h(s))$ (T' is as in Lemma 2). But this simply means that $h(s)$ is P-integrable and $P(h) = (P)\int h(s)\,ds$.

5. We are now in a position to generalize Theorem 2 of §30 in [5]. Let G be a locally compact group; consider its group algebra $L^1(G)$. As it was defined above, a positive A-functional on $L^1(G)$ is a linear mapping $\varphi: L^1(G) \to A$ which has an extension φ' to

$$L^1(G)e = \{\lambda e + a \mid a \in L^1(G), \lambda \text{ complex}\}$$

and the extension φ' is a positive A-functional on $L^1(G)e$. Positive definite A-function was defined in [7] as a mapping $p: G \to \tau(A)$ such that $\sum_{i,j} a_i^* p(t_i^{-1}t_j)a_j \geqq 0$ for $t_i \in G$, $a_i \in A$.

THEOREM 2. *If $p: G \to \tau(A)$ is a continuous (with respect to the norm ( )) positive definite A-function then the mapping $t \to x(t)p(t)$ is both Pettis and P-integrable for each $x \in L^1(G)$ and the function $\varphi(x) = \int x(t)p(t)\,dt$ is a positive A-functional on $L^1(G)$. Conversely each positive A-functional on $L^1(G)$ is of this form.*

PROOF. Let p be a continuous positive definite A-function on G. According to Theorem 1 of [7] there exists a continuous representation $t \to U_t$ of G by A-unitary operators on a Hilbert module H and a member f_0 of H such that $p(t) = (U_t f_0, f_0)$ for all $t \in G$. But then the mapping $t \to V_t = U_t^*$ is also a continuous representation of G by A-unitary operators and so it follows from Theorem 2 of [7] that the mapping $x \to Tx = \int_G x(t)V_t\,dt$ is a regular *-representation of $L^1(G)$.

Now note that the mapping $t \to x(t)V_t f$ is Pettis integrable for each $f \in H$ and $(P) \int_G x(t)V_t f\, dt = (\int x(t)V_t\, dt)f$ for all $f \in H$ (the last integral here has the same meaning as the corresponding integral in Theorem 2 of [7]). It follows then from Lemma 3 above that

$$\begin{aligned}(f_0, Txf_0) &= \left(f_0, \left(\int_G x(t)V_t\, dt\right) f_0\right) = (f_0, P(x(t)V_t f_0)) \\ &= (P)\int_G x(t)(f_0, V_t f_0)\, dt = (P)\int_G x(t)(U_t f_0, f_0)\, dt \\ &= (P)\int_G x(t)p(t)\, dt = \varphi(x)\end{aligned}$$

for all $x \in L^1(G)$. If we extend T to $L^1(G)e$ by setting $T(\lambda e + x) = \lambda I + Tx$ we see immediately that φ has an extension φ' defined by $\varphi'(\lambda e + x) = (f_0, T(\lambda e + x)f_0) = \lambda(f_0, f_0) + (f_0, Txf_0) = \lambda(f_0, f_0) + \varphi(x)$ and that φ' is a positive A-functional on $L^1(G)e$.

The converse is also established in a similar manner. If φ is a positive A-functional on $L^1(G)$ then φ is of the form $\varphi(x) = (f_0, Txf_0)$ for some *-representation T of $L^1(G)$ by A-linear operators. It follows from Theorem 2 of [7] that there exists a continuous representation $t \to U_t$ of G (by A-unitary operators) such that $Tx = \int_G x(t)U_t\, dt$ for all $x \in L^1(G)$. Then $p(t) = (U_t f_0, f_0)$ is a positive A-function and $\varphi(t) = (f_0, Txf_0) = \int_G x(t)(f_0, U_t f_0)\, dt = \int_G x(t)p(t)$ for all $x \in L^1(G)$.

References

1. W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. **57** (1945), 364–386. MR **7**, 126.

2. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, rev. ed., Amer. Math. Soc., Providence, R.I., 1957. MR **19**, 664.

3. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953. MR **14**, 883.

4. G. W. Mackey, *Commutative Banach algebra*, Lecture notes, Harvard University, 1952; Livraria Castelo, Rio de Janeiro, 1959.

5. M. A. Naimark, *Normed rings*, GITTL, Moscow, 1956; English transl., Noordhoff, Groningen, 1959. MR **19**, 870; **22** #1824.

6. P. P. Saworotnow, *A generalized Hilbert space*, Duke Math. J. **35** (1968), 191–197. MR **37** #3333.

7. ———, *Representation of a topological group on a Hilbert module*, Duke Math. J. **37** (1970), 145–150. MR **41** #7442.

8. P. P. Saworotnow and J. C. Friedell, *Trace-class for an arbitrary H*-algebra*, Proc. Amer. Math. Soc. **26** (1970), 95–100.

9. P. P. Saworotnow, *Trace-class and centralizer of an H*-algebra*, Proc. Amer. Math. Soc. **26** (1970), 101–104.

Department of Mathematics, The Catholic University of America, Washington D.C. 20017