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INVARIANT MEANS ON LOCALLY COMPACT SEMIGROUPS

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ABSTRACT. Let G be a locally compact semigroup (jointly continuous semigroup operation), $M(G)$ the algebra of all bounded regular Borel measures on G (with convolution as multiplication), E a separated locally convex space and S a compact convex subset of E. We show that there is a left invariant mean on the space LUC(G) of all bounded left uniformly continuous functions on G iff G has the following fixed point property: For any bilinear mapping $T:M(G)\times E\to E$ (denoted by $(\mu,s)\to T_\mu(s)$) such that (a) $T_\mu(S)\subset S$ for any $\mu\geqq 0$, $\|\mu\|=1$, (b) $T_{\mu*\nu}=T_\mu\circ T_\nu$ for any μ, $\nu\in M(G)$, (c) $T_\mu:S\to S$ is continuous for any $\mu\geqq 0$, $\|\mu\|=1$, and (d) $\mu\to T_\mu(s)$ is continuous for each $s\in S$ when $M(G)$ has the topology induced by the seminorms $p_f(\mu)=|\int f\,d\mu|$, $f\in \mathrm{LUC}(G)$, there is some $s_0\in S$ such that $T_\mu(s_0)=s_0$ for any $\mu\geqq 0$, $\|\mu\|=1$.

1. **Introduction.** The purpose of this paper is to extend certain results in the theory of invariant means on locally compact groups to locally compact semigroups. Let G be a locally compact group with measure algebra $M(G)$. The present author has proved in [**17**, Theorem 3.3] that $L_\infty(G)$[2] has a topological left invariant mean iff G has the fixed point property on convex compacta for separately continuous actions of $M(G)$. In this paper, we shall prove among other things, that if G is a locally compact semigroup, then there is a left invariant mean on the space of all bounded left uniformly continuous functions on G iff G has a similar fixed point property which turns out to be equivalent to the above fixed point property when G is a locally compact group.

2. **Terminologies.**

2.1 *The measure algebra $M(G)$.* A locally compact semigroup is a semigroup with a locally compact topology for which the semigroup

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[2] $L_\infty(G)$ is defined with respect to a fixed left Haar measure (see Hewitt and Ross [**9**, Definition 12.11]). For definition of topological left invariant mean, consult [**8**] or [**16**].

operation is *jointly* continuous. Let G be a locally compact semigroup and $M(G)$ the Banach space of all bounded regular Borel measures on G [**9**, Definition 14.9]. It is known that $M(G)$ is a convolution algebra if we define the convolution $\mu * \nu$ of two measures μ, ν in $M(G)$ by the formula

$$\int f(z)\, d(\mu * \nu)(z) = \iint f(xy)\, d\mu(x)\, d\nu(y)$$

for any $f \in C_0(G)$, the continuous functions on G which vanish at infinity (see Glicksberg [**6**]). It follows immediately that the same formula is valid for any $f \in L_1(G, |\mu| * |\nu|)$ ($|\mu|$ denotes the total variation of μ; see Hewitt and Ross [**9**, Theorems 14.6 and 14.14]). This can be proved by repeating "mutatis mutandis" the arguments used in Hewitt and Ross [**9**, Theorem 19.10], where G is assumed to be a locally compact group (because only the continuity of the map $(x, y) \to xy$ of $G \times G$ into G is invoked in their proof). Let $M_0(G)$ be the set of all probability measures in $M(G)$ ($\mu \in M_0(G)$ iff $\mu \geqq 0$ and $\|\mu\| = 1$). It is easy to see that $M_0(G)$ is a convolution semigroup.

2.2 *The space* LUC(G) *and its dual.* Let CB(G) be the Banach space of all continuous bounded functions on the locally compact semigroup G. A function $f \in \mathrm{CB}(G)$ is called left uniformly continuous[3] if the map $a \to l_a f$ of G into CB(G) is norm continuous ($l_a f(x) = f(ax)$, $x \in G$). The space of all left uniformly continuous functions on G is denoted by LUC(G). It is known that LUC(G) is a translation invariant linear subspace of CB(G) containing the constants and is left introverted in the sense that if $f \in \mathrm{LUC}(G)$, $m \in \mathrm{LUC}(G)^*$, then $m_l(f) \in \mathrm{LUC}(G)$ where $m_l(f)$ is defined by $m_l(f)(x) = m(l_x f)$, $x \in G$. (See Namioka [**12**] or Mitchell [**11**].) The space LUC(G)* can be made into a Banach algebra if we define the Arens product $m \odot n$ of two functionals m, n in LUC(G)* by $m \odot n(f) = m(n_l(f))$ for any $f \in \mathrm{LUC}(G)$.

For each $f \in \mathrm{LUC}(G)$, define a seminorm [**14**, Chapter I, §4] p_f on the linear space $M(G)$ by $p_f(\mu) = |\int f\, d\mu|$, $\mu \in M(G)$. The locally convex topology on $M(G)$ determined by these seminorms [**14**, Theorem 3, p. 15] is denoted by τ. Note that each $\mu \in M(G)$ can be regarded as a functional in LUC(G)* if we set $\bar{\mu}(f) = \int f\, d\mu$, $f \in \mathrm{LUC}(G)$. (But this embedding might not be one to one, in other words, τ might not be separated. If G is a locally compact group, then $C_0(G) \subset \mathrm{LUC}(G)$. Hence τ is separated and therefore the same as the w^* topology of LUC(G)* restricted to $M(G)$.)

A functional m in LUC(G)* is called a mean iff $\|m\| = m(1) = 1$. A mean m is left invariant if $m(l_a f) = m(f)$ for any $a \in G, f \in \mathrm{LUC}(G)$.

[3] Some authors call these functions right uniformly continuous and denote them by UCB$_r$(G) (see for example Greenleaf [**8**]).

3. **A lemma.** We shall need the following lemma which gives some useful information about the spaces $M(G)$, $\mathrm{LUC}(G)$ and its dual $\mathrm{LUC}(G)^*$. It is also of independent interest.

LEMMA 3.1. *Let G be a locally compact semigroup, then*:

(a) *For each μ in $M(G)$, the map $m \to \bar{\mu} \odot m$ is w^*-w^* continuous on any norm bounded subset of* $\mathrm{LUC}(G)^*$.

(b) *For each m in* $\mathrm{LUC}(G)^*$, *the map $\mu \to \bar{\mu} \odot m$ is continuous with respect to the topology τ of $M(G)$ and the w^* topology of* $\mathrm{LUC}(G)^*$.

(c) *If $\mu, \nu \in M(G)$, then $\bar{\mu} \odot \bar{\nu} = \overline{\mu * \nu}$ on* $\mathrm{LUC}(G)$.

(d) *A mean m in* $\mathrm{LUC}(G)^*$ *is left invariant iff $\bar{\mu} \odot m = m$ for any μ in $M_0(G)$.*

(e) $\mathrm{LUC}(G)$ *has a left invariant mean iff there is a net μ_α in $M_0(G)$ such that $\mu * \mu_\alpha - \mu_\alpha \xrightarrow{\tau} 0$ in $M(G)$ for any μ in $M_0(G)$.*

PROOF. (a) Let $m_\alpha \to m$ in weak* topology of $\mathrm{LUC}(G)^*$ and $\|m_\alpha\|$, $\|m\| \leqq K$. For each f in $\mathrm{LUC}(G)$, $s, t \in G$,

$$|(m_\alpha)_l f(s) - (m_\alpha)_l f(t)| = |m_\alpha(l_s f) - m_\alpha(l_t f)| \leqq K \cdot \|l_s f - l_t f\|.$$

Therefore the family of functions $(m_\alpha)_l f$ is equicontinuous (Kelley [**10**, p. 232]). Since $(m_\alpha)_l f \to m_l(f)$ pointwise on G, the convergence is uniform on every compact subset of G (Kelley [**10**, Theorem 7.15]). Let μ in $M(G)$ have compact support, then $\bar{\mu} \odot m_\alpha(f) - \bar{\mu} \odot m(f) = \int (m_\alpha)_l f \, d\mu - \int m_l(f) \, d\mu \to 0$.

Since the measures in $M(G)$ with compact supports are norm dense in $M(G)$ and $\|(m_\alpha)_l f\| \leqq K \cdot \|f\|$, it follows that $\bar{\mu} \odot m_\alpha \to \bar{\mu} \odot m$ in w^* topology of $\mathrm{LUC}(G)^*$, for any μ in $M(G)$.

(b) Let m in $\mathrm{LUC}(G)^*$ be fixed. Suppose $\mu_\alpha \to \mu$ in the topology τ of $M(G)$, then $\bar{\mu}_\alpha \odot m(f) - \bar{\mu} \odot m(f) = \int m_l(f) \, d\mu_\alpha - \int m_l(f) \, d\mu \to 0$ since $m_l(f) \in \mathrm{LUC}(G)$ for any $f \in \mathrm{LUC}(G)$. Therefore $\bar{\mu}_\alpha \odot m \to \bar{\mu} \odot m$ in weak* topology of $\mathrm{LUC}(G)^*$.

(c) Let $\mu, \nu \in M(G)$, then

$$\begin{aligned} \bar{\mu} \odot \bar{\nu}(f) = \bar{\mu}(\bar{\nu}_l(f)) &= \int \bar{\nu}_l(f)(x) \, d\mu(x) = \int \bar{\nu}(l_x f) \, d\mu(x) \\ &= \iint f(xy) \, d\mu(x) \, d\nu(y) = \int f(z) \, d(\mu * \nu)(z) = \overline{\mu * \nu}(f), \end{aligned}$$

$f \in \mathrm{LUC}(G)$. (Recall the remarks in §2.1.) Hence

$$\bar{\mu} \odot \bar{\nu} = \overline{\mu * \nu}$$

on $\mathrm{LUC}(G)$.

(d) If $m \in \mathrm{LUC}(G)^*$ is such that $\bar{\mu} \odot m = m$ for any $\mu \in M_0(G)$, then clearly m is left invariant since $M_0(G)$ contains the point measures. But the convex combinations of these point measures are w^* dense in the set of means in $\mathrm{LUC}(G)^*$ (more precisely, it is their images under the map $\mu \to \bar{\mu}$ which are w^* dense). Therefore if m is left invariant and $\mu \in M_0(G)$, let μ_α be a net of convex combinations of point measures such that $\bar{\mu}_\alpha \xrightarrow{w^*} \bar{\mu}$ in $\mathrm{LUC}(G)^*$, then $\mu_\alpha \xrightarrow{\tau} \mu$ in $M(G)$ and hence $\bar{\mu}_\alpha \odot m \xrightarrow{w^*} \bar{\mu} \odot m$ in $\mathrm{LUC}(G)^*$ by (*b*). Now clearly $\bar{\mu}_\alpha \odot m = m$ since m is left invariant. Consequently $\bar{\mu} \odot m = m$.

(e) Suppose $\mathrm{LUC}(G)$ has a left invariant mean m, then by (d), $\bar{\mu} \odot m = m$ for any $\mu \in M_0(G)$ which is w^* dense in the set of means in $\mathrm{LUC}(G)^*$. Let $\mu_\alpha \in M_0(G)$ be a net such that $\bar{\mu}_\alpha \xrightarrow{w^*} m$ in $\mathrm{LUC}(G)^*$. Since $\|\bar{\mu}_\alpha\| \leqq \|\mu_\alpha\| = 1$ and $\|m\| = 1$, by (a), $\bar{\mu} \odot \bar{\mu}_\alpha \xrightarrow{w^*} \bar{\mu} \odot m$ in $\mathrm{LUC}(G)^*$. Therefore

$$\overline{\mu * \mu_\alpha - \mu_\alpha} = \overline{\mu * \mu_\alpha} - \bar{\mu}_\alpha = \bar{\mu} \odot \bar{\mu}_\alpha - \bar{\mu}_\alpha \xrightarrow{w^*} \bar{\mu} \odot m - m = 0$$

in $\mathrm{LUC}(G)^*$ by (c). In other words $\mu * \mu_\alpha - \mu_\alpha \xrightarrow{\tau} 0$ in $M(G)$.

Conversely assume that for some net μ_α in $M_0(G)$, $\mu * \mu_\alpha - \mu_\alpha \xrightarrow{\tau} 0$ for any $\mu \in M_0(G)$. By w^* compactness of the set of means in $\mathrm{LUC}(G)^*$, we can assume $\bar{\mu}_\alpha \xrightarrow{w^*} m$ for some mean m in $\mathrm{LUC}(G)^*$ (passing to a subnet if necessary). Then

$$\begin{aligned}\bar{\mu} \odot m - m &= \bar{\mu} \odot \left(w^* \lim_\alpha \bar{\mu}_\alpha\right) - w^* \lim_\alpha \bar{\mu}_\alpha \\ &= w^* \lim_\alpha (\bar{\mu} \odot \bar{\mu}_\alpha - \bar{\mu}_\alpha) = w^* \lim_\alpha (\overline{\mu * \mu_\alpha - \mu_\alpha}) = 0\end{aligned}$$

by (a) and (c). Hence m is a left invariant mean on $\mathrm{LUC}(G)$.

4. **Main theorems.**

DEFINITION 4.1. Let G be a locally compact semigroup, E a separated locally convex space. An action T of $M(G)$ on E is a homomorphism of $M(G)$ into the algebra of linear operators in E. Thus we have a bilinear map $T{:}M(G) \times E \to E$ (where $(\mu, s) \to T_\mu(s)$, $\mu \in M(G)$, $s \in E$) such that $T_{\mu * \nu} = T_\mu \circ T_\nu$ for any $\mu, \nu \in M(G)$. If S is a compact convex subset of E, we say that S is $M_0(G)$-invariant under T if $T_\mu(S) \subset S$ for any $\mu \in M_0(G)$. In this case T induces an action $T{:}M_0(G) \times S \to S$ of the convolution semigroup $M_0(G)$ on S as affine maps in S (see **[17]** for definition of actions in the case when G is a locally compact group).

THEOREM 4.2. *Let G be a locally compact semigroup, then the following conditions are equivalent*:

(a) $\mathrm{LUC}(G)$ *has a left invariant mean.*

(b) *If* $T{:}M(G) \times E \to E$ *is any action of* $M(G)$ *on a separated locally convex space E and S any compact convex* $M_0(G)$*-invariant subset of E such that* (i) *for each* $\mu \in M_0(G)$, $T_\mu{:}S \to S$ *is continuous and* (ii) *for each*

$s \in S$, the map $\mu \to T_\mu(s)$ from $M(G)$ into E is continuous when $M(G)$ has the topology τ, then the induced action $T:M_0(G) \times S \to S$ has a fixed point.

PROOF. Assume that LUC(G) has a left invariant mean. By Lemma 3.1(e), there is a net $\mu_\alpha \in M_0(G)$ such that $\mu * \mu_\alpha - \mu_\alpha \xrightarrow{\tau} 0$ in $M(G)$. Let $T:M(G) \times E \to E$ be any action of $M(G)$ on E and $S \subset E$ a compact convex $M_0(G)$-invariant subset of E satisfying conditions (i) and (ii) of (b). Consider the net $T_{\mu_\alpha}(s)$ in S where $s \in S$ is arbitrary but fixed. By compactness of S, we can assume $T_{\mu_\alpha}(s) \to s_0$ in S (use a subnet if necessary). We shall show that s_0 is the required fixed point of the action $T:M_0(G) \times S \to S$ by repeating the arguments used in the proof of **[17,** Theorem 3.1]. Let $\mu \in M_0(G)$, then

$$T_\mu(s_0) = T_\mu(\lim_\alpha T_{\mu_\alpha}(s)) = \lim_\alpha T_\mu(T_{\mu_\alpha}(s)) = \lim_\alpha T_{\mu * \mu_\alpha}(s)$$
$$= \lim_\alpha \{T_{\mu * \mu_\alpha - \mu_\alpha}(s) + T_{\mu_\alpha}(s)\} = \lim_\alpha T_{\mu_\alpha}(s) = s_0$$

by condition (i), linearity of $\mu \to T_\mu(s)$, condition (ii) and the fact that $\mu * \mu_\alpha - \mu_\alpha \xrightarrow{\tau} 0$ in $M(G)$.

Conversely, assume (b). Let $E = \text{LUC}(G)^*$ with w^* topology and $S =$ the set of means in LUC(G)*. Define an action of $M(G)$ on E by $T_\mu(m) = \bar{\mu} \odot m$ for each $\mu \in M(G)$, $m \in \text{LUC}(G)^*$. It is clear that the map $T:M(G) \times E \to E$ is bilinear. To show that T is an action, suppose $\mu, \nu \in M(G)$, then

$$T_{\mu * \nu}(m) = \overline{\mu * \nu} \odot m = (\bar{\mu} \odot \bar{\nu}) \odot m = \bar{\mu} \odot (\bar{\nu} \odot m) = T_\mu(T_\nu(m))$$

by Lemma 3.1(c) and associativity of the Arens product. Hence $T_{\mu * \nu} = T_\mu \circ T_\nu$. Obviously S is a w^* compact convex subset of LUC(G)*. If $m \in S$, $\mu \in M_0(G)$ then $\|T_\mu(m)\| \leqq \|\mu\| \cdot \|m\| = 1$ and $T_\mu(m)(1) = (\bar{\mu} \odot m)(1) = 1$. Consequently $\|T_\mu(m)\| = T_\mu(m)(1) = 1$ and $T_\mu(m)$ is a mean on LUC(G). Thus S is $M_0(G)$-invariant under T. By Lemma 3.1(a) and (b), it is straightforward to verify that the action T defined above satisfies the continuity conditions (i) and (ii) of (b). Therefore by assumption (b), the induced action $T:M_0(G) \times S \to S$ must have a fixed point which is a left invariant mean on LUC(G) by Lemma 3.1(d). This completes the proof of the theorem.

REMARK 4.3. Suppose $T:M(G) \times E \to E$ is any action of $M(G)$ on E and S a compact convex $M_0(G)$-invariant subset of E satisfying the continuity conditions (i) and (ii) of (b) in the theorem. Let $T:G \times S \to S$ be defined by $T_x(s) = T_{\mu(x)}(s)$ where $x \in G$, $s \in S$ and $\mu(x)$ is the point measure at the point x. T is an action of G as affine maps in S (that is, each map $T_x:S \to S$ is affine and $T_{xy} = T_x \circ T_y$ for any $x, y \in G$). Moreover, the map $(x, s) \to T_x(s)$ is separately continuous. For if $x_\alpha \to x$ in

G, then $\mu(x_\alpha) \xrightarrow{\tau} \mu(x)$ in $M(G)$ and hence $T_{x_\alpha}(s) = T_{\mu(x)_\alpha}(s) \to T_{\mu(x)}(s) = T_x(s)$ in S by (ii) while continuity of the map $s \to T_x(s)$ follows from (i). In general, this is all we can say about the action $T:G \times S \to S$ of G. However if G is a locally compact *group*, then the same action, being separately continuous, is also jointly continuous by a theorem of Ellis [**5**, Theorem 1]. Consequently, if we assume that LUC(G) has a left invariant mean, then the action $T:G \times S \to S$ must have a fixed point by Rickert's theorem [**13**, Theorem 4.2]; see also Mitchell [**11**, Theorem 2] for a more general result. It follows that s_0 is also a fixed point of the induced action $T:M_0(G) \times S \to S$ (because the convex combinations of point measures are w^* dense in the set of means in LUC(G)*). This gives yet another proof of (a) implies (b) in the case when G is a locally compact group.

5. **Special cases.**

5.1 *Locally compact group.* Let G be a locally compact group. It was proved in [**17**] that $L_\infty(G)$ has a topological left invariant mean (see [**16**] for definition) iff G has the following fixed point property:

(*) For any action $T:M(G) \times E \to E$ of $M(G)$ on a separated locally convex space E and any compact convex $M_0(G)$-invariant subset S of E such that the map $M(G) \times E \to E$ is separately continuous when $M(G)$ has the norm topology, the induced action $T:M_0(G) \times S \to S$ has a fixed point.

Now it is known that $L_\infty(G)$ has a topological left invariant mean iff LUC(G) has a left invariant mean (see Greenleaf [**8**] where LUC(G) is denoted by $\mathrm{UCB}_r(G)$). Therefore we have the following theorem.

THEOREM 5.2. *Let G be a locally compact group, then the following are equivalent*:

(a) $L_\infty(G)$ *has a topological left invariant mean.*

(b) LUC(G) *has a left invariant mean.*

(c) *G has fixed point property* (b) *of Theorem* 4.2.

(d) *G has fixed point property* (*) *of* 5.1.

5.3 *Compact semigroups.* Suppose that G is a compact semigroup. It is well known that CB(G) = LUC(G) (Namioka [**12**]) and that CB(G) has a left invariant mean iff the kernel $K(G)$ of G is a compact group and the unique invariant mean is the Haar integral over $K(G)$ (see Rosen [**15**] or Glicksberg and de Leeuw [7]). Let ν be the normalised Haar measure of $K(G)$, define $\lambda \in M_0(G)$ by $\int f\, d\lambda = \int_{K(G)} f|_{K(G)}\, d\nu$, $f \in C_0(G) = \mathrm{CB}(G)$. By direct calculation, one shows that $\int f\, d\mu * \lambda = \int f\, d\lambda$ for any $\mu \in M_0(G)$ and $f \in C_0(G)$. Hence $\mu * \lambda = \lambda$. Now if $T:M(G) \times E \to E$ is any action of $M(G)$ on E and $S \subset E$ is compact convex and $M_0(G)$-invariant, then $T_\mu(T_\lambda(s)) = T_{\mu*\lambda}(s) = T_\lambda(s)$ for any μ in $M_0(G)$ and $s \in S$. In other words, $T_\lambda(s)$ is a fixed point of the induced action $T:M_0(G) \times S \to S$, for each

$s \in S$, without any continuity conditions on the action $T:M(G) \times E \to E$ whatsoever.

Finally, it is also interesting to note that when G is compact, the mapping $\mu \to \bar{\mu}$ is precisely the natural isometric isomorphism $M(G) = C_0(G)^*$, and the Arens product $\odot$ is nothing but convolution of measures in $M(G)$.

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