PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 2, February 1972

SHORTER NOTES

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A HOLOMORPHIC FUNCTION HAVING A DISCONTINUOUS INVERSE

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ABSTRACT. An example is given of a function f which is holomorphic in the open unit ball of l^∞ and which extends in a natural way to the closed unit ball. The extended function gives a one-to-one correspondence of the closed ball with itself but the inverse function fails to be continuous on the image of the open ball under the map f.

Let X and Y be Banach spaces and let D be open $D \subset X$. It is unknown whether the assumption: $f: D \to Y$ is an injective holomorphic map (Fréchet differentiable [1, Chapters 3 and 26]) of D onto an open set $f(D) \subset Y$ implies f^{-1} is holomorphic. In this note we give an example which is related to this problem.

Let $a=(a_1, a_2, \cdots) \in l^\infty$ and assume $|a_k|<1$ for each k. Let B and $\bar{B}$ be the open and closed unit ball respectively in l^∞. Define $f_a: B \to \bar{B}$ by $f_a(x)=(w_1, w_2, \cdots)$ where $w_k=(x_k-a_k)/(1-\bar{a}_k x_k)$, $k=1, 2, \cdots$. Clearly f_a extends to $\bar{B}$ to give a one-to-one correspondence of $\bar{B}$ with $\bar{B}$. Also,

$$Df_a(x)(y) = \left(\frac{1-|a_1|^2}{(1-\bar{a}_1 x_1)^2}\, y_1, \frac{1-|a_2|^2}{(1-\bar{a}_2 x_2)^2}\, y_2, \cdots\right)$$

so that $Df_a(x)$ is a bounded linear map for each $x \in B$ and f_a is holomorphic in B.

Now choose $a_k=k/(k+1)$ (any choice of a_k satisfying $|a_k|<1$ and $\sup|a_k|=1$ will do). Then $f_a(0)=-a$ and $\|f_a(0)\|=1$. By the maximum principle [1, Theorem 3.13.1], $\|f_a(x)\|=1$ for all $x \in B$. If $|x_k|=1$ for some k then $\|f_a(x)\|=1$ and if for some subsequence $\{x_{k_j}\}$ we have $|x_{k_j}-a_{k_j}| \geqq \varepsilon > 0$ then $\|f_a(x)\|=1$. Hence $\|f_a(x)\|<1$ implies $|x_k|<1$ for all k and

Received by the editors June 14, 1971.
AMS 1970 *subject classifications*. Primary 58C10.
Key words and phrases. Holomorphic function, holomorphic inverse, Banach space.

$\lim_{k\to\infty} x_k = 1$. However, f_a cannot be continuous at such an x for we may choose y so that $y_k = x_k$ if $k \neq k_0$ and $y_{k_0} = 1$ so that $\|y - x\| = |1 - x_{k_0}|$ which can be made arbitrarily small while $\|f_a(y) - f_a(x)\| \geqq 1 - \|f_a(x)\|$ which is positive and constant. We also find $f_a^{-1}(x) = f_{-a}(x)$ so f_a is discontinuous on $f_a^{-1}(B)$ and f_a^{-1} is discontinuous on $f_a(B)$.

References

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