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C-EMBEDDED SUBSETS OF PRODUCTS

N. NOBLE

ABSTRACT. It is shown that each dense subset of R^n is z-embedded, from which it follows that a dense subset is C-embedded if and only if it is G_δ-dense. These results extend to, for example, all products of separable metric spaces.

All spaces are assumed to be completely regular Hausdorff; R denotes the real line and vX the Hewitt realcompactification of X. Recall that a subset X of Y is *z-embedded* in Y if each zero set of X is the intersection of X with some zero set of Y. A subset of Y is *G_δ-dense* in Y if it meets each nonempty G_δ-set of Y, and the *G_δ-closure* of a subset is the largest subspace in which it is G_δ-dense.

THEOREM 1. *For X a dense subset of R^n, the following conditions are equivalent*:

(i) *Some superset of X in R^n is vX*;
(ii) *the G_δ-closure of X in R^n is vX*;
(iii) *the G_δ-closure of X in R^n is realcompact.*

COROLLARY. *For $X \subseteq R^n$, $vX = R$ if and only if X is G_δ-dense.*

Theorem 1 follows immediately from the fact that each space is G_δ-dense in its Hewitt realcompactification but is not G_δ-dense in any larger space, a theorem of Hager and Johnson that a G_δ-dense subset is C-embedded if and only if it is z-embedded [**2**, Proposition 3] and the following:

THEOREM 2. *Each dense subspace of R^n is z-embedded.*

PROOF. Let X be a dense subspace of R^n and let Z be a zero set in X, say $Z = \bigcap_n U_n$ where each U_n is open and contains the closure in X of U_{n+1}. Let F_n be the closure of U_n in R^n; then $F_n \cap X = \mathrm{cl}(U_n)$ so for $F = \bigcap_n F_n$, $F \cap X = Z$. Thus it suffices to show that F is a zero set.

Since X is dense, each F_n is the closure of its interior, so by [**6**, Theorem 3] each F_n has the form $\pi_n^{-1}(H_n)$ where π_n is the projection onto some

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countable subproduct and H_n is a closed subspace of that subproduct. It follows that F also has this form, say $F=\pi^{-1}(H)$. But like any closed subset of $R^{\aleph_0}$, H is a zero set. Therefore F is a zero set, as desired.

Notice by the same proof, Theorem 2 holds with $R^{\mathfrak{n}}$ replaced by any product space Y satisfying:

(i) Each finite subproduct of Y (and hence Y itself [**5**, Corollary 1.4]) satisfies the countable chain condition, so the structure theorem for regular closed sets holds [**5**, Proposition 2.2].

(ii) Each finite subproduct of Y (and hence each countable subproduct of Y, by [**3**, Proposition 2.1]) is perfect, i.e., has each closed subset a G_δ.

(iii) Each countable subproduct of Y has each closed G_δ a zero set.

In particular, Theorems 1 and 2 hold with $R^{\mathfrak{n}}$ replaced by any product of separable metric spaces. Regarding further generalizations, note that if X and Y are pseudocompact subsets of βN which contain N and for which $X\times Y$ is not pseudocompact, then $X\times Y$ is G_δ-dense in $\beta N\times\beta N$ but is not z-embedded (since if it were it would be C^*-embedded which, by Glicksberg's Theorem, is the case only if $X\times Y$ is pseudocompact). Theorem 1 will be applied in [**4**] to characterize spaces Y for which $C(Y)$ is realcompact in various standard function space topologies.

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Canary Road, Westlake, Oregon 97493