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MONOTONICITY OF POSITIVE SEMIDEFINITE HERMITIAN MATRICES[1]

RUSSELL MERRIS AND STEPHEN PIERCE

ABSTRACT. Inequalities which compare elements of the convex cone of positive semidefinite hermitian matrices with products of roots of elements are proved. They yield inequalities for Schur functions (generalized matrix functions) which, when specialized to the determinant, give a result of R. Bellman and L. Mirsky.

Introduction. The following result was proved in [**2**] and [**7**]. (See also [**1**, p. 63] and [**4**, p. 115].) *Let $\theta_1, \cdots, \theta_k$ be positive numbers such that $\theta_1+\cdots+\theta_k=1$. Let $A_1, \cdots, A_k$ be positive definite hermitian n-square matrices. Then*

$$\det\left(\sum_{i=1}^{k}\theta_i A_i\right) \geqq \prod_{i=1}^{k}(\det(A_i))^{\theta_i}$$

with equality if and only if $A_1=\cdots=A_k$.

In this note, we observe that the above result generalizes in several directions (Theorems 2, 3, and 4). Theorem 1 provides the case of equality in several of our inequalities.

When A is positive semidefinite (positive definite) hermitian, we write $A\geqq 0$ ($A>0$). If $A\geqq 0$, $B\geqq 0$, $A-B\geqq 0$, we write $A\geqq B$.

Results. Let H be a subgroup of S_n, the symmetric group. Let χ be a character on H. Define $d(A)$, which depends on H and χ, by

$$d(A)=\sum_{\sigma\in H}\chi(\sigma)\prod_{t=1}^{n}a_{t\sigma(t)}$$

where $A=(a_{ij})$. These functions were studied by Schur [**8**]. If $H=S_n$ and $\chi=\text{sgn}$, then $d=\det$. If $H=S_n$ and $\chi\equiv 1$, $d=\text{per}$ (permanent).

We will frequently use the result in [**6**] that $A\geqq B$ implies $d(A)\geqq d(B)$.

THEOREM 1. *Suppose $\chi\equiv 1$ or $d=\det$. If $A\geqq B$ and $d(A)=d(B)\neq 0$, then $A=B$.*

We will prove Theorem 1 later.

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THEOREM 2. *Let $A_i \geqq 0$, $i=1, \cdots, k$, and suppose the A_i are pairwise commuting. If $\theta_1, \cdots, \theta_k$ are positive numbers such that $\theta_1+\cdots+\theta_k=1$, then*

$$\sum_{i=1}^{k} \theta_i A_i \geqq \prod_{i=1}^{k} A_i^{\theta_i}$$

with equality if and only if $A_1=\cdots=A_k$. Thus,

$$d\left(\sum_{i=1}^{k} \theta_i A_i\right) \geqq d\left(\prod_{i=1}^{k} A_i^{\theta_i}\right). \tag{1}$$

If $\chi\equiv 1$ or $d=\det$, equality holds in (1) *if and only if $A_1=\cdots=A_k$.*

PROOF. Since the A_i's commute, they can be simultaneously diagonalized. The result then follows from the arithmetic-geometric mean inequality and Theorem 1.

THEOREM 3. *Let $A_1, \cdots, A_k$ be as in Theorem 2. Then,*

$$d\left(\left(\sum_{i=1}^{k} \theta_i A_i\right)^{1/n}\right) \geqq \prod_{i=1}^{k} (d(A_i^{1/n}))^{\theta_i}. \tag{2}$$

PROOF. By the extended Minkowski inequality [3],

$$d\left(\left(\sum_{i=1}^{k} \theta_i A_i\right)^{1/n}\right) \geqq \sum_{i=1}^{k} \theta_i\, d(A_i^{1/n}).$$

The arithmetic-geometric mean inequality now yields (2).

Without assuming commutativity, it is difficult to obtain much information on Schur functions other than the determinant. Generalizing Mirsky's method, we can prove

THEOREM 4. *Let $A \geqq 0$, $B=C^*C>0$. Let $0<\theta<1$. Then*

$$\theta A + (1-\theta)B \geqq C^*(C^{*-1}AC^{-1})^{\theta}C$$

with equality if and only if $A=B$. Thus

$$d(\theta A + (1-\theta)B) \geqq d(C^*(C^{*-1}AC^{-1})^{\theta}C). \tag{3}$$

If $d=\det$ or $\chi\equiv 1$, equality holds in (3) *if and only if $A=B$.*

PROOF. For $\lambda \geqq 0$, $\theta\lambda+1-\theta \geqq \lambda^{\theta}$ with equality if and only if $\lambda=1$. Thus, for any $H \geqq 0$,

$$\theta H + (1-\theta)I \geqq H^{\theta}$$

with equality if and only if $H=I$. Take $H=C^{*-1}AC^{-1}$.

The Bellman-Mirsky result follows from any of Theorems 2, 3, or 4, using the multiplicativity of the determinant.

PROOF OF THEOREM 1. Let $\lambda_1 \geqq \cdots \geqq \lambda_n \geqq 0$, $\mu_1 \geqq \cdots \geqq \mu_n \geqq 0$ be the eigenvalues of A and B respectively. A standard argument using the Courant-Fischer minimax theorem proves that $A \geqq B$ implies $\lambda_i \geqq \mu_i$, $i=1, \cdots, n$. Thus if $A \geqq B$ and $\det(A)=\det(B)\neq 0$, $\lambda_i=\mu_i$, $i=1, \cdots, n$. Hence trace $(A-B)=0$, and $A=B$.

If $\chi\equiv 1$, our proof requires the machinery of the associated transformation.

Let V be the inner product space of column n-tuples. Let $E=\{e_1, \cdots, e_n\}$ be the standard basis, i.e., e_j has a 1 in position j and 0 elsewhere. Let $\otimes V$ be the nth tensor power of V. If $x_1, \cdots, x_n \in V$, write $x_1\otimes\cdots\otimes x_n$ as their tensor product. For $\sigma\in S_n$, let $P(\sigma^{-1})$ be the linear operator defined by

$$P(\sigma^{-1})x_1 \otimes \cdots \otimes x_n = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}, \qquad x_i \in V.$$

Set

$$T(H, \chi) = \frac{1}{h} \sum_{\sigma\in H} \chi(\sigma)P(\sigma),$$

where h is the order of H. Write W for the range of $T(H, \chi)$. Let

$$x_1 * \cdots * x_n = T(H, \chi)x_1 \otimes \cdots \otimes x_n.$$

Let $\otimes A$ denote the nth Kronecker power of A, i.e.,

$$(\otimes A)x_1 \otimes \cdots \otimes x_n = Ax_1 \otimes \cdots \otimes Ax_n.$$

The space W is invariant under $\otimes A$ since $T(H, \chi)$ and $\otimes A$ commute. So, let $K(A)$ be the restriction of $\otimes A$ to W.

The inner product $(\cdot, \cdot)$ on V induces an inner product on $\otimes V$ which satisfies

$$(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n) = \prod_{t=1}^{n} (x_t, y_t).$$

It has been proved, [**5**, p. 318], that there is a set Δ of integer sequences of length n such that the set

$$\{e_\alpha^* = e_{\alpha_1} * \cdots * e_{\alpha_n} : \alpha = (\alpha_1, \cdots, \alpha_n) \in \Delta\}$$

forms an orthogonal basis of W. Moreover, we may assume $(1, \cdots, n)\in\Delta$.

For each $\alpha\in\Delta$, let k_α be the inverse of the norm of e_α^*. Then

$$E^* = \{k_\alpha e_\alpha^* : \alpha \in \Delta\}$$

is an orthonormal basis of W. (When $\alpha=(1, \cdots, n)$, $k_\alpha=h^{1/2}$.) Suppose E^* is ordered arbitrarily. Let $\boldsymbol{K}(A)$ be the matrix representation of $K(A)$ with respect to E^*. It is known that if $A \geqq 0$, so is $\boldsymbol{K}(A)$, and $A \geqq B$ implies $\boldsymbol{K}(A) \geqq \boldsymbol{K}(B)$ (see [**6**]).

Now one can easily verify that the main diagonal entry in $\boldsymbol{K}(A)$ corresponding to the sequence $(1, \cdots, n)$ is

$$\begin{aligned} &h(K(A)e_1 * \cdots * e_n, e_1 * \cdots * e_n) \\ &\quad = h(Ae_1 * \cdots * Ae_n, e_1 * \cdots * e_n) = d(A). \end{aligned}$$

If $A \geqq B$ and $d(A)=d(B)$ then, being $\geqq 0$, $\boldsymbol{K}(A)-\boldsymbol{K}(B)$ has a zero row and column corresponding to $(1, \cdots, n)$. It follows that

$$Ae_1 * \cdots * Ae_n = Be_1 * \cdots * Be_n. \tag{4}$$

For $\chi \equiv 1$, we can apply Lemma 2.4 of [**5**] to (4) to see that $A=BQ$, where Q is a monomial whose entry in column j is d_j. Since $d(A)=d(B)\neq 0$, we have $\prod_j d_j=1$. Since $B \geqq 0$, $|b_{ij}|^2 \leqq b_{ii}b_{jj}$ for all pairs i, j. If $\sigma \in S_n$ is the permutation corresponding to Q, then $a_{jj}=b_{\sigma(j)j}\, d_j \leqq b^{1/2}_{\sigma(j)\sigma(j)} b^{1/2}_{jj} |d_j|$. Hence, $\prod_j a_{jj}=\prod_j b_{jj}$. But, $A \geqq B$ implies $a_{jj} \geqq b_{jj} \geqq 0$. Also, since $d(B)\neq 0$, no row of B can be zero, and hence $b_{jj}\neq 0$ for all j. Therefore $a_{jj}=b_{jj}$, $1 \leqq j \leqq n$, or $\operatorname{trace}(A-B)=0$, and the result is established.

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National Bureau of Standards, Washington, D.C. 20234

Current address (*Merris*): Department of Mathematics, California State College, Hayward, California 94542

Current address (*Pierce*): Department of Mathematics, University of Toronto, Toronto 181, Ontario, Canada