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A COMPACTIFICATION OF LOCALLY COMPACT SPACES

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ABSTRACT. Every locally compact space X has its topology determined by its 1-1 compact images and hence has a compactification ξX obtained as the closure of the natural embedding of X in the product of those images, just as the Stone-Čech compactification βX can be obtained by embedding X in a product of intervals. The obvious question is whether $\xi X=\beta X$. In this paper we prove that $\xi X=\beta X$ if X either is 0-dimensional or contains an arc, and give an example in which $\xi X\neq\beta X$.

Preliminaries. All maps are continuous, and all compact and locally compact spaces are Hausdorff. For any space X, let $\mathscr{K}(X)$ denote the set of all 1-1 maps of X onto a compact space, let $Y=\prod\{f(X)\mid f\in\mathscr{K}(X)\}$ and let $e:X\to Y$ be the evaluation map; if e is a homeomorphism then $\mathrm{cl}_Y\, e(X)$ is a compactification of X which we denote by ξX. If X is locally compact then e is necessarily a homeomorphism because for any closed $F\subseteq X$ there is an $f\in\mathscr{K}(X)$ such that $f(F)$ is closed in $f(X)$: choose $x\in F$, let X' be the set X with the topology consisting of all open $U\subseteq X$ such that either $x\notin U$ or $X-U$ is compact, and let $f:X\to X'$ be the natural map. Note that, by a standard argument, ξX is the smallest compactification of X to which every $f\in\mathscr{K}(X)$ can be extended.

PROPOSITION. *Suppose that for any two disjoint zero-sets Z_1 and Z_2 of a locally compact space X there is a map f from X into a compact subspace Y of X such that $f(Z_1)$ and $f(Z_2)$ have disjoint closures. Then $\xi X=\beta X$.*

PROOF. Let $\tilde{e}:\beta X\to\xi X$ be the Stone extension of the embedding $e:X\to\xi X$. Since e is a homeomorphism it follows that $\tilde{e}(\beta X-X)=\xi X-e(X)$; hence we need only show that $\tilde{e}(p)\neq\tilde{e}(q)$ for any two distinct $p,q\in\beta X-X$. Let A_p and A_q be the free z-ultrafilters on X converging to p and q, respectively, choose disjoint $Z_1\in A_p$ and $Z_2\in A_q$, and let f and Y be as hypothesized. Then let $\tilde{f}:\beta X\to Y$ be the Stone extension of f, let $Y'=\tilde{f}(\beta X-X)$, let $X'=(\beta X-X)\cup Y'$, and define $g:X'\to Y'$ by requiring that

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$g(x)$ be $\tilde{f}(x)$ or x according as $x \in \beta X - X$ or $x \in Y'$. Since $\beta X - X$ and Y' are disjoint closed subsets of βX, g is continuous. Therefore, since X' is compact and Y' is Hausdorff, $\mathscr{D}' = \{g^{-1}(y) \mid y \in Y'\}$ is a closed, upper semicontinuous decomposition of X', i.e., $\bigcup \{D \in \mathscr{D}' \mid D \cap F \neq \varnothing\}$ is closed for every closed $F \subseteq X'$. As a consequence, if $\mathscr{D} = \mathscr{D}' \cup \{\{x\} \mid x \in X - Y'\}$ then, for any closed $F \subseteq \beta X$,

$$\bigcup \{D \in \mathscr{D} \mid D \cap F \neq \varnothing\} = F \cup (\bigcup \{D \in \mathscr{D}' \mid D \cap (F \cap X') \neq \varnothing\})$$

is closed in βX, so that $\mathscr{D}$ is a closed, upper semicontinuous decomposition of βX. Thus, if h is the projection of βX onto the quotient space determined by $\mathscr{D}$, then $h(\beta X)$ is Hausdorff and hence compact. Let $k = h|X$. Then $k \in \mathscr{K}(X)$ so that there is a map $\hat{k}: \xi X \to k(X)$ such that $\hat{k} \circ e = k$. Now $Z_1 \in A_p$ so that $p \in \mathrm{cl}_{\beta X} Z_1$ and hence $g(p) = \tilde{f}(p) \in \mathrm{cl} f(Z_1)$. Similarly, $g(q) \in \mathrm{cl} f(Z_2)$. Therefore, since $\mathrm{cl} f(Z_1)$ and $\mathrm{cl} f(Z_2)$ are disjoint, it follows that $g(p) \neq g(q)$ and hence, by the definition of h, that $h(p) \neq h(q)$. But $\hat{k} \circ \tilde{e}$ and h agree on the dense subset X of βX so that $\hat{k} \circ \tilde{e} = h$. Hence $\tilde{e}(p) \neq \tilde{e}(q)$, as required.

COROLLARY 1. *If a locally compact space is* 0*-dimensional in the sense of* [**3**], *then* $\xi X = \beta X$.

PROOF. Any two disjoint zero-sets Z_1 and Z_2 of X are contained in disjoint open sets U_1 and U_2 whose union is X. Choose $x_i \in U_i$, let $Y = \{x_1, x_2\}$, and define $f: X \to Y$ by requiring $f(x) = x_i$ if $x \in U_i$.

COROLLARY 2. *If a locally compact space X contains an arc, then* $\xi X = \beta X$.

PROOF. By assumption, there is a map $g: [0, 1] \to X$ such that $g(0) \neq g(1)$. For any two disjoint zero-sets Z_1 and Z_2 of X, there is a map $h: X \to [0, 1]$ such that $h(Z_1) = 0$ and $h(Z_2) = 1$. Let $f = g \circ h$.

EXAMPLE. According to Cook [**1**] there is a nontrivial, compact, connected space which admits no map into itself other than the identity map and the constant maps. Let C be such a space and let α be the first ordinal with card $\alpha >$ card C. Note that necessarily card α is uncountable. Now let x_1 and x_2 be distinct points of C, let p be a point not in C, and set $H = C \times [0, \alpha] \times [0, \alpha]$ and $K = \{p\} \times [0, \alpha] \times [0, \alpha]$. Then

$$A = (\{x_1\} \times \{\alpha\} \times [0, \alpha]) \cup (\{x_2\} \times [0, \alpha] \times \{\alpha\})$$

is closed in H and the map $\theta: A \to K$ defined by requiring $f(\langle x, \beta, \gamma \rangle) = \langle p, \beta, \gamma \rangle$ is continuous, so that [**2,** VI.6.1 and VII.3.4] the space $Y = H \cup_\theta K$ obtained by "attaching H to K by θ" is compact. Let Y' be the free union of H and K and let $\psi: Y' \to Y$ be the quotient map. Let

$$X' = Y' - ((C \cup \{p\}) \times \{\alpha\} \times \{\alpha\})$$

and let $X=\psi(X')$. Then $Y-X=\psi(Y'-X')$ is closed in Y, and hence X is locally compact. Moreover, X is dense in Y so that, in order to show that $\beta X=Y$, it suffices to show that any $f\in C^*(X)$ can be extended to $\tilde{f}\in C^*(Y)$. But $g=f\circ(\psi|X')\in C^*(X')$ so that, by standard techniques [3, 8L and 9K], one can show that there is a $\beta<\alpha$ such that g is constant on $X'\cap(\{x\}\times[\beta,\alpha]\times[\beta,\alpha])$ for all $x\in C\cup\{p\}$. Hence g can be extended to $\tilde{g}\in C^*(Y')$ by setting $\tilde{g}(\langle x,\alpha,\alpha\rangle)=g(\langle x,\beta,\beta\rangle)$. Moreover, $\tilde{g}(\langle x_1,\alpha,\alpha\rangle)=g(\langle x_1,\beta,\beta\rangle)=g(\langle x_1,\alpha,\beta\rangle)=g(\langle p,\alpha,\beta\rangle)=g(\langle p,\beta,\beta\rangle)=\tilde{g}(\langle p,\alpha,\alpha\rangle)$ and, similarly, $\tilde{g}(\langle x_2,\alpha,\alpha\rangle)=\tilde{g}(\langle p,\alpha,\alpha\rangle)$ so that there is an $\tilde{f}\in C^*(Y)$ such that $\tilde{f}\circ\psi=\tilde{g}$, the continuity of $\tilde{f}$ following from the fact that ψ is a quotient map. Clearly $\tilde{f}$ is the desired extension of f so that $\beta X=Y$ as asserted.

Now consider any $f\in\mathscr{K}(X)$, let $\tilde{f}\colon Y\to f(X)$ be the Stone extension of f, and let $D=f^{-1}(\tilde{f}(Y-X))$. Then D is a closed subset of X with card $D\leqq$ card $(Y-X)=$card $C<$card α, so that $\psi^{-1}(D)$ is a closed subset of X' of cardinality less than card α and hence, by a standard argument, compact. Thus D is compact and hence a continuous image of $Y-X$ under the map $f^{-1}\circ\tilde{f}$. But $Y-X$ is just C with the points x_1 and x_2 identified, so that D is a continuous image of C under a map which is not 1-1. Therefore, since every component of X is either a singleton or a homeomorph of C, it follows that D consists of a single point. Thus $\tilde{f}(Y-X)$ is a singleton so that f can be extended to the one-point compactification of X obtained by identifying $Y-X$ into a point. Hence ξX is just the one-point compactification of X; in particular, $\xi X\neq\beta X$.

REMARK. The obvious open problem is to find an internal characterization of the locally compact spaces X for which $\xi X=\beta X$.

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