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REGULAR COMPACTIFICATIONS OF CONVERGENCE SPACES

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Abstract. This note gives a simple characterization for the class of convergence spaces for which regular compactifications exist and shows that each such convergence space has a largest regular compactification.

Introduction. It has been shown by Wyler [5] that for every Hausdorff convergence space S there is a regular (including Hausdorff) compact convergence space S^* and a continuous map $j: S \to S^*$ with the following property: for every continuous map $f: S \to T$, where T is regular and compact, there is a unique continuous map $g: S^* \to T$ such that $f = g \circ j$. Richardson [4] obtained a similar result, but with the following important distinctions: (1) the compactification space S^* is Hausdorff but not necessarily regular (for convergence spaces, Hausdorff plus compact does not imply regular); (2) the map j is a dense embedding. But there is in general no largest Hausdorff compactification, and indeed the number of distinct maximal Hausdorff compactifications can be quite large.

The conclusions of both [4] and [5] suggest that a more satisfactory compactification theory for convergence spaces might result from an investigation of regular compactifications, although it is known (see [2]) that there are regular convergence spaces which cannot be embedded in any compact regular space. What we obtain in this note is a characterization of the class of convergence spaces for which regular compactifications exist, and we show that each such convergence space has a largest regular compactification.

1. For basic information about convergence spaces the reader is asked to refer to [1] and [2]. If there is no possibility of confusion, a convergence space (S, q) will be denoted simply by S. A space is *regular* if it is Hausdorff and has the property: $\mathscr{F}$ converges to x implies that the closure of $\mathscr{F}$ (denoted $\mathrm{cl}\, \mathscr{F}$) converges to x. We shall denote by $\mathrm{cl}_S A$ the closure of a subset A of a convergence space S. The *pretopological modification* πS of a convergence space (S, q) is the space (S, p), where p is the finest pretopology

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on S coarser than q; the *topological modification* λS is defined analogously. Recall that $\pi S = \lambda S$ iff the closure operator cl_S is idempotent.

If A is a subset of a convergence space S, then the subspace defined by A (also denoted by A) is given as follows: A filter $\mathscr{F}$ on A A-converges to x in A iff the filter on S generated by $\mathscr{F}$ S-converges to x.

PROPOSITION 1. *The closure operator for a compact regular convergence space is idempotent.*

PROOF. Let S be compact and regular, A a subset of S. By the theorem of [**4**], the identity map from A into S has a continuous extension f to A^*, the Stone-Čech compactification of A. Since $f(\mathrm{cl}_{A^*} A) = f(A^*) \subseteq \mathrm{cl}_S A$ by continuity of f and $f(A^*)$ is compact and hence S-closed, it follows that $\mathrm{cl}_S A$ is closed.

PROPOSITION 2. *If A is a subspace of S, then πA is the subspace of πS determined by A.*

PROPOSITION 3. *If A is a subspace of a compact regular convergence space, then πA is Hausdorff and topological.*

PROOF. Let S be a compact regular convergence space containing A. By Proposition 2 it suffices to show that πS is Hausdorff and topological; that it is topological follows from Proposition 1. To see that πS is Hausdorff, let $\mathscr{F}$ be an ultrafilter in πS which converges to both x and y. By compactness $\mathscr{F}$ converges in S to some point z and regularity guarantees that $\mathrm{cl}_S \mathscr{F}$ also converges in S to z. But each neighborhood of x is in $\mathscr{F}$, so x is in each member of $\mathrm{cl}_S \mathscr{F}$ and hence the Hausdorffness of S implies that $x=z$. Similarly, $y=z$, and so $x=y$.

2. A *compactification* (T,f) of a convergence space S consists of a compact convergence space T along with a dense embedding f of S into T. For a different definition, see §7 of [**3**].

If (T,f) is a compactification of S, then it is a simple matter to verify that $(\pi T, f)$ is a compactification of πS. From this fact and Proposition 3 we deduce the next result.

PROPOSITION 4. *If (T,f) is a regular compactification of S, then $(\pi T, f)$ is a Hausdorff (topological) compactification of πS.*

THEOREM 1. *A regular convergence space S has a regular compactification iff πS is a completely regular topological space and each ultrafilter which is finer than the neighborhood filter at x S-converges to x for all x in S.*

PROOF. Assume the given conditions. Then πS has a topological compactification (T,f). Let T_1 be the convergence space consisting of the set T equipped with the finest convergence structure r on T which satisfies the following conditions: if $f(S)$ belongs to $\mathscr{G}$, then $\mathrm{cl}_T \mathscr{G}$ r-converges to x

in $f(S)$ iff $f^{-1}(\mathscr{G})$ S-converges to $f^{-1}(x)$; if $\mathscr{F}$ is an ultrafilter such that $T-f(S)$ belongs to $\mathscr{F}$, then $\mathrm{cl}_T(\mathscr{F})$ r-converges to x in $f(S)$ iff $\mathscr{F}$ T-converges to x; if $\mathscr{H}$ is an ultrafilter on T, then $\mathrm{cl}_T\,\mathscr{H}$ r-converges to y in $T-f(S)$ iff $\mathscr{H}$ T-converges to y.

By this construction, it is clear that T_1 and T coincide relative to ultrafilter convergence, and so the closure operators for these spaces coincide. The fact that T_1 is regular can be established with the aid of the following lemma: If $\mathscr{F}$ is an ultrafilter on $T-f(S)$ which T-converges to x in $f(S)$, then there is an ultrafilter $\mathscr{G}$ on S which S-converges to $f^{-1}(x)$ such that $f^{-1}(\mathrm{cl}_T\,\mathscr{F})\geqq\mathrm{cl}_S\,\mathscr{G}$. Finally, it is easy to establish that $f{:}\,S\to T_1$ is a dense embedding.

If S has a regular compactification (T,f), then πS is a completely regular topological space by Proposition 3. To show that S has the second property, let $\mathscr{F}$ be an ultrafilter finer than the S-neighborhood filter at x. Let y be the point in T to which $f(\mathscr{F})$ T-converges. Since $f(\mathscr{F})$ is finer than the neighborhood filter for $f(x)$, $\dot{f}(x)\geqq\mathrm{cl}_T\,f(\mathscr{F})$, and so necessarily $y=f(x)$. Thus $f^{-1}(f(\mathscr{F}))=\mathscr{F}$ S-converges to x since f is an embedding.

A regular compactification T of a convergence space S is a *Stone-Čech regular compactification* if each continuous function from S into a compact regular space has a continuous extension to T.

THEOREM 2. *If a convergence space has a regular compactification, then it has a Stone-Čech regular compactification.*

PROOF. Let S be a convergence space with a regular compactification; let (T,f) be the (topological) Stone-Čech compactification of $\pi(S)$, and let (T_1,f) be the regular compactification of S constructed above. If g is any continuous function from S into a compact regular space R, then $g{:}\,\pi S\to\pi R$ has a unique continuous extension $h{:}\,T\to\pi R$ such that $h\circ f=g$, and it follows easily that $h{:}\,T_1\to R$ is also continuous.

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