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A GENERALIZATION OF MORI'S THEOREM

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ABSTRACT. In this article, we consider a generalization of Mori's theorem which is: Let R be a Zariski ring; if the completion of R is a unique factorization domain, then so is R.

Mori's theorem states that a Zariski ring R is a unique factorization domain (UFD) if its completion $\hat{R}$ is a UFD. Validity of the theorem stems from the facts that R is a Gelfand ring, a filtered ring whose radical is open, and that $\hat{R}$ is a faithfully flat R-module. From this point of view, a generalization of Mori's theorem is studied in this paper. We prove that if R is a Gelfand ring whose completion $\hat{R}$ is a flat R-module and R is a pure submodule of $\hat{R}$, in particular, $\hat{R}$ is a faithfully flat R-module, then R is a UFD whenever $\hat{R}$ is a UFD. Applying the result, we also consider a generalization of Nagata-Mori's theorem [5].

In this paper, every ring is assumed to be a commutative ring with identity. A filtered ring R with a filtration $\{q_n; n=0, 1, 2, 3, \cdots\}$ will be denoted by (R, q_n), and its completion by $(\hat{R}, \hat{q}_n)$, where $q_0=R$ and $\hat{q}_n$ is the completion of q_n. We say that R is hat-flat if $\hat{R}$ is a flat R-module (cf. [1]).

DEFINITION 1. Let E be a module over a ring R, and F a submodule of E. F will be called *a pure submodule* of E if $rE\cap F=rF$ for all $r\in R$.

It is obvious that if the completion $\hat{R}$ of a filtered ring R is a faithfully flat R-module, then R is a pure submodule of $\hat{R}$.

DEFINITION 2. A ring with a linear topology is called a *Gelfand ring* if its radical is open (cf. [3, p. 44]).

Note that a hat flat Gelfand ring is necessarily a separated topological ring by [3, Corollary (5.5)]. Evidently, a filtered ring (R, q_n) is a Gelfand ring if and only if $q_1\subseteq \operatorname{rad}(R)$. Thus every Zariski ring R is a hat-flat Gelfand ring which is a pure submodule of $\hat{R}$ because $\hat{R}$ is a faithfully flat R-module.

The following lemma is easy to verify.

LEMMA. *Let B be an ideal of a filtered ring (R, q_n). Then B is derse in $\hat{R}B$ for the $(\hat{q}_nB)$-topology.*

THEOREM 1. *Let (R, q_n) be a Gelfand ring which satisfies the following conditions*: (i) *R is hat-flat and* (ii) *R is a pure submodule of the R-module $\hat{R}$. Then R is a UFD, if $\hat{R}$ is a UFD.*

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PROOF. Firstly, R satisfies the ascending chain condition for principal ideals because $\hat{R}$ does and R is a pure submodule of $\hat{R}$. Next, let $a, b \in R$ and $P = Ra \cap Rb$. Since R is hat flat, we have that $\hat{R}P = \hat{R}a \cap \hat{R}b$ by [2, p. 32, Proposition 6]; moreover, $\hat{R}P = \hat{R}c$ for some $c \in \hat{R}P$ as $\hat{R}$ is a UFD. According to the Lemma, there exists a $c_1 \in P$ such that $c \equiv c_1 \bmod \hat{q}_1 P$, hence $\hat{R}c = \hat{R}c_1 + \hat{q}_1 P$. Now put $\hat{R}c/\hat{R}c_1 = \hat{R}P/\hat{R}c_1 = E$, then E is a finitely generated $\hat{R}$-module and $\hat{q}_1 E = E$. By [3, Proposition (5.1)], $\hat{R}$ is a Gelfand ring, so that $\hat{q}_1 \subseteq \operatorname{rad}(\hat{R})$. Applying Nakayama's lemma we have $E = (0)$, that is, $\hat{R}c = \hat{R}c_1$. It follows that $P = Ra \cap Rb = Rc_1$ as R is a pure submodule of $\hat{R}$. Thus the intersection of any two principal ideals of R is principal, which implies that R is a UFD.

COROLLARY 1. *Let R be a Gelfand ring such that $\hat{R}$ is a faithfully flat R-module. Then R is a UFD, if $\hat{R}$ is a UFD.*

The method of the proof of Theorem 1 is the same as the proof of Mori's theorem in [5, p. 2]. We demonstrated that the method still works for the rings with conditions a bit weaker than Zariski rings. Simultaneously, we have shown that Theorem 1, as well as its Corollary 1, is a generalization of Mori's theorem.

COROLLARY 2. *Let (R, q_n) be a Noetherian filtered ring satisfying either one of the following two conditions:*

(i) *R is a hat-flat Gelfand ring,*

(ii) *the (q_n)-topology of R is stronger than its radical topology and $\hat{R}$ is a Noetherian ring. Then R is a UFD, if $\hat{R}$ is a UFD.*

PROOF. For case (i), the corollary follows from Corollary 1 to Theorem 1 and [3, Proposition (5.4)].

For case (ii), it is a result of Corollary 1 to Theorem 1 and [4, Proposition (5.4)].

DEFINITION 3. A topological ring is said to be *topologically artinian* if it is equipped with a linear topology and there exists a fundamental system of neighborhoods of 0 consisting of ideals of finite length.

PROPOSITION. *A Gelfand ring (R, q_n) which is topologically artinian is necessarily a quasi-semilocal ring.*

PROOF. Since R is a Gelfand ring, $q_1 \subseteq \operatorname{rad}(R)$. Moreover, R/q_1 is an artinian ring, hence there exist only a finite number of maximal ideals in R/q_1. Now, we can conclude that there exist only a finite number of maximal ideals in R, because every maximal ideal of R contains $q_1 \subseteq \operatorname{rad}(R)$.

THEOREM 2. *Let R be a hat-flat Gelfand ring which is topologically artinian. Then R is a UFD, if $\hat{R}$ is a UFD.*

PROOF. The radical of R is open as R is a Gelfand ring, whence every maximal ideal of R is open. Consequently, $\hat{R}$ is a faithfully flat R-module due to [**1**, Proposition 5]. Hence the proposition follows immediately from Corollary 1 to Theorem 1.

In the following Theorem 3 we consider a generalization of Nagata-Mori's theorem [**6**, p. 56, Theorem 10].

THEOREM 3. *Let (R, q_n) be a hat-flat ring which is either a Noetherian ring, or a separated topologically artinian ring, satisfying the ascending chain condition (a.c.c.) for principal ideals. If $\hat{R}$ is a UFD and $S=1+q_1$ is generated by prime elements of R, then R is also a UFD.*

PROOF. Suppose that R is a Noetherian hat-flat ring, and put $R'=S^{-1}R$. Then R' is a Gelfand ring and $\hat{R}'=\hat{R}$ by Theorem (5.1) and Corollary (5.2) both of [**3**]. Since $\hat{R}$ is a flat R-module, it is a flat R'-module by [**3**, Proposition (5.6)]. Thus R' is a Noetherian hat-flat Gelfand ring. Since $\hat{R}=\hat{R}'$ is a UFD, R' is also a UFD by Corollary 2 to Theorem 1. Clearly, R is an integral domain which satisfies the a.c.c. for principal ideals, hence R is a UFD by Nagata's theorem [**5**, Lemma 1.7]. Next, we assume the other case for R. Clearly, $(R', S^{-1}q_n)$ is a Gelfand ring which is topologically artinian, and $R \subseteq R' \subseteq \hat{R}$ because $(\hat{R}, \hat{q}_n)$ is a Gelfand ring. This implies that $\hat{R}'=\hat{R}$ since $(R', S^{-1}q_n)$ is a subspace of $(\hat{R}, \hat{q}_n)$ due to the fact that R is separated (cf. [**6**, p. 55, the proof of remark]). Moreover R' is hat-flat by [**2**, Proposition 6]. Thus R' is a hat-flat Gelfand ring which is topologically artinian. According to Theorem 2, R' is a UFD. Now, that R is a UFD also follows from Nagata's theorem.[1]

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[1] In Nagata's theorem the ring need not be Noetherian if it satisfies the a.c.c. for principal ideals.