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ALGEBRAIC ALGEBRAS WITH INVOLUTION

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ABSTRACT. The following theorem is proved: Let R be an algebra with involution over an uncountable field F. Then if the symmetric elements of R are algebraic, R is algebraic.

In this paper we consider the following question:

"Let R be an algebra with involution over a field F, and assume that the symmetric elements S of R are algebraic over F. Is R algebraic over F?"

Previous results related to this question have been obtained by restricting the kind of algebraic relationships satisfied by the symmetric elements. For example, it was shown by Baxter and Martindale [**1**] for fields of characteristic not 2, and later by the author [**5**] for arbitrary fields, that if the symmetric elements are algebraic of bounded degree (or more generally, satisfy a polynomial identity), then R must be algebraic. Another such result concerns rings whose symmetric elements are periodic (that is, for each $s \in S$, there is some integer $n(s) > 1$ such that $s^{n(s)} = s$). In this case, the author has shown [**6**], [**7**] that R must be algebraic; in fact R satisfies a polynomial identity. When R is a division ring, much more can be said: I. N. Herstein and the author [**2**] have shown that R must actually be commutative. Finally, it has been shown by Osborn [**8**] that if S is nil and F is uncountable, then R is nil. This answers for uncountable fields a question of McCrimmon [**4**, p. 391]:

"If S is nil, is R nil?"

An affirmative answer to this question in general would follow from an affirmative answer to the first question. For, as has been observed by both McCrimmon [**4**, p. 390] and Osborn [**8**, p. 306], if S is nil then R must be a radical ring. But if R is algebraic, every element of the radical is nil; thus R would be nil.

The result presented here differs from those described above in that no additional restrictions are imposed on the symmetric elements. We prove:

THEOREM. *Let R be an algebra with involution over an uncountable field F. Then if the symmetric elements of R are algebraic, R is algebraic.*

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If R is a ring, an involution on R is simply an anti-automorphism of period 2. By an algebra with involution, we mean that R has an involution * as a ring, and that the field F has an automorphism $\alpha \to \bar{\alpha}$ of period 2 such that $(\alpha r)^* = \bar{\alpha} r^*$, for all $\alpha \in F$, $r \in R$. $S = \{x \in R \mid x^* = x\}$ will denote the symmetric elements of R.

LEMMA 1. *Let R be an algebra with unit over a field F, and say $x \in R$ with $x^2 = rx + s$, $r, s \in R$. Let $A = \left(\begin{smallmatrix} 0 & 1 \\ s & r \end{smallmatrix}\right)$, the 2×2 matrix. Then if A is algebraic over F, x is algebraic over F.*

PROOF. We first notice that $A^2 = rA + sI$, where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Thus

$$A^3 = (rA + sI)A = r(rA + sI) + sA = (r^2 + s)A + rsI = r_1 A + s_1 I.$$

Similarly, $A^n = r_{n-2}A + s_{n-2}I$, where r_{n-2}, $s_{n-2} \in R$ for all $n > 2$. Since x satisfies $x^2 = rx + s$, by the same procedure as for A we find that also $x^n = r_{n-2}x + s_{n-2}$, for all $n > 2$. Now if A is algebraic over F, there exists some polynomial $p(\lambda) \in F[\lambda]$ such that $p(A) = 0$. We claim that $p(x) = 0$.

For, if $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$, $\alpha_i \in F$, then

$$\begin{aligned} 0 &= p(A) \\ &= (r_{n-2}A + s_{n-2}I) + \alpha_{n-1}(r_{n-3}A + s_{n-3}I) \\ &\quad + \cdots + \alpha_2(rA + sI) + \alpha_1 A + \alpha_0 I \\ &= (r_{n-2} + \alpha_{n-1}r_{n-3} + \cdots + \alpha_2 r + \alpha_1 1)A \\ &\quad + (s_{n-2} + \alpha_{n-1}s_{n-3} + \cdots + \alpha_2 s + \alpha_0 1)I \\ &= tA + t'I, \qquad \text{where } t, t' \in R. \end{aligned}$$

But

$$tA + t'I = \begin{pmatrix} 0 & t \\ ts & tr \end{pmatrix} + \begin{pmatrix} t' & 0 \\ 0 & t' \end{pmatrix} = \begin{pmatrix} t' & t \\ ts & t' + tr \end{pmatrix},$$

so $tA + t'I = 0$ implies $t = 0$ and $t' = 0$. Since $x^i = r_{i-2}x + s_{i-2}$, $i > 2$, $p(x) = tx + t' = 0$ and thus x is algebraic.

Recall that if R is any algebra with unit, we may consider R as an algebra of linear transformations by letting R act on itself by right multiplication. Thus a characteristic root (or vector) of an element $r \in R$ will mean a characteristic root (or vector) of r considered as a linear transformation acting by right multiplication. For any $r \in R$, we also define the spectrum $\sigma(r) = \{\alpha \in F$ such that $r - \alpha \cdot 1$ has no inverse in $R\}$. The resolvent $\rho(r)$ is the complement of $\sigma(r)$ in F.

LEMMA 2. *Let R be an algebra with involution over any field F such that S is algebraic. Assume that R has a unit element, and that F is fixed element-wise by *. Choose $x \in R$, and let $A = \left(\begin{smallmatrix} 0 & 1 \\ s & r \end{smallmatrix}\right)$, where $r = x + x^*$ and $s = -x^*x$.*

Consider r acting by right multiplication on R, and A acting by right multiplication on the 2×2 matrices over R. Then for any $\alpha\in\rho(r)$, the resolvent of r, with $\alpha\neq 0$, either α is a characteristic root of A or $\alpha\in\rho(A)$, the resolvent of A.

PROOF. Let $a=\left(\begin{smallmatrix}0&1\\0&r\end{smallmatrix}\right)$ and $y=\left(\begin{smallmatrix}0&0\\s&0\end{smallmatrix}\right)$, so $A=a+y$. Let $\alpha\in F$, $\alpha\neq 0$. Assume that $a-\alpha I$ is invertible in R_2. Then there is some matrix $\left(\begin{smallmatrix}b&c\\d&e\end{smallmatrix}\right)$, $b, c, d, e\in R$, such that $(a-\alpha I)\left(\begin{smallmatrix}b&c\\d&e\end{smallmatrix}\right)=I$. Since $a-\alpha I=\left(\begin{smallmatrix}-\alpha&1\\0&r-\alpha\end{smallmatrix}\right)$, this gives

$$\begin{pmatrix}-\alpha & 1\\ 0 & r-\alpha\end{pmatrix}\begin{pmatrix}b & c\\ d & e\end{pmatrix}=\begin{pmatrix}-\alpha b+d & -\alpha c+e\\ (r-\alpha)d & (r-\alpha)e\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

In particular, $(r-\alpha)e=1$ and so $(r-\alpha)^{-1}$ exists in R. Also $d=0$, $b=-1/\alpha$, and $c=(r-\alpha)^{-1}$, and thus

$$(a-\alpha I)^{-1}=\begin{pmatrix}-\dfrac{1}{\alpha} & \dfrac{1}{\alpha}(r-\alpha)^{-1}\\ 0 & (r-\alpha)^{-1}\end{pmatrix}.$$

Certainly if $\alpha\neq 0$ and $(r-\alpha)^{-1}$ exists in R, we have that $(a-\alpha I)^{-1}$ exists in R_2 by the expression for $(a-\alpha I)^{-1}$.

To summarize: if $\alpha\neq 0$, $a-\alpha I$ is invertible if and only if $r-\alpha$ is invertible. This means that $\rho(a)\subseteq\rho(r)$; in fact if $0\notin\rho(r)$, $\rho(a)=\rho(r)$, and if $0\in\rho(r)$, $\rho(r)=\rho(a)\cup\{0\}$.

Choose $\alpha\in\rho(r)$, $\alpha\neq 0$. Consider $A-\alpha I=(a-\alpha I)+y$. Multiplying on the right by $(a-\alpha I)^{-1}$ gives $I+y(a-\alpha I)^{-1}=I+y'$, where $y'=y(a-\alpha I)^{-1}$. We claim that y' is algebraic. For,

$$y'=\begin{pmatrix}0&0\\s&0\end{pmatrix}\begin{pmatrix}-\dfrac{1}{\alpha} & \dfrac{1}{\alpha}(r-\alpha)^{-1}\\ 0 & (r-\alpha)^{-1}\end{pmatrix}=\begin{pmatrix}0 & 0\\ -\dfrac{s}{\alpha} & \dfrac{s}{\alpha}(r-\alpha)^{-1}\end{pmatrix}.$$

Now $(s/\alpha)(r-\alpha)^{-1}=-(x^*x/\alpha)(r-\alpha)^{-1}$. Since $r-\alpha\in S$, $(r-\alpha)^{-1}\in S$ and thus $x(-(r-\alpha)^{-1}/\alpha)x^*\in S$. But then $x(-(r-\alpha)^{-1}/\alpha)x^*$ is algebraic, and so $x^*x(-(r-\alpha)^{-1}/\alpha)=(s/\alpha)(r-\alpha)^{-1}$ is algebraic. Say $t((s/\alpha)(r-\alpha)^{-1})=0$, some polynomial $t(\lambda)$. We may assume that $t(\lambda)$ has no constant term (multiply by λ if necessary). Thus $t(y')\in\left(\begin{smallmatrix}0&0\\R&0\end{smallmatrix}\right)$, so $t(y')^2=0$, and y' is algebraic.

For an algebraic element, the spectrum coincides with the characteristic roots of the linear transformation [3, p. 246]. Hence either $-1\in\rho(y')$ or -1 is a characteristic root of y'.

If $-1\in\rho(y')$, then $(I+y')^{-1}$ exists, and so

$$((a+y)-\alpha I)^{-1}=(a-\alpha I)^{-1}(I+y')^{-1}$$

and $\alpha\in\rho(A)$. But if -1 is a characteristic root of y', then there is an $x\neq 0$, $x\in R_2$, so that $x(I+y')=0$. Then $x(a+y-\alpha I)=x(I+y')(a-\alpha I)=0$, and α is a characteristic root of A.

PROOF OF THE THEOREM. Choose $x\in R$. Then $x^2-(x+x^*)x+x^*x=0$; letting $r=x+x^*$ and $s=-x^*x$, we have $x^2=rx+s$. Thus by Lemma 1, it is enough to show that $A=\left(\begin{smallmatrix}0 & 1\\ s & r\end{smallmatrix}\right)$ is algebraic.

First note that we may assume that F is left elementwise fixed by the automorphism $-$. For if not, let F_0 be the subfield of F fixed elementwise by $-$. R is certainly an algebra over F_0, F_0 is uncountable, and F is algebraic over F_0 (as "$-$" has period 2). Thus if $s\in S$ is algebraic over F, s is algebraic over F_0. Thus R satisfies the hypotheses as an algebra over F_0. But if R is algebraic over F_0, R is certainly algebraic over F.

We may also assume that R contains a unit element. For if not, consider the algebra $R_1=\{(r,\alpha)\mid r\in R, \alpha\in F\}$, where addition is defined componentwise and multiplication is given by $(r,\alpha)\cdot(t,\beta)=(rt+\alpha t+\beta r, \alpha\beta)$. R_1 has an involution, given by $(r,\alpha)^*=(r^*,\alpha)$, and is an algebra over F by $\alpha(r,\beta)=(\alpha r, \alpha\beta)$. Now the symmetric elements of R_1 are algebraic: let (s,α) be a symmetric element. Since $s\in S$, s is algebraic over F, say $p(s)=0$, where $p(\lambda)\in F[\lambda]$. Then (s,α) satisfies the polynomial $p(\lambda-\alpha)$, so is algebraic. Certainly if R_1 is algebraic, R is algebraic.

Finally, we may assume that R is finitely generated over F—if not, replace R by $R'=F[1,x,x^*]$. This means that the dimension of R over F is countable.

We apply Lemma 2 to see that for any $\alpha\in\rho(r)$, either α is a characteristic root of A or $\alpha\in\rho(A)$. But $\rho(r)$ is uncountable, since the spectrum of r consists of the roots of its minimal polynomial [**3**, p. 20], and so either $\rho(A)$ is uncountable or the set of distinct characteristic roots of A is uncountable. The latter is impossible, for then R_2, the 2×2 matrices, would contain an uncountable number of characteristic vectors, which are linearly independent. This contradicts the dimension of R_2 over F being countable.

Thus it must be that $\rho(A)$ is uncountable, and so A is algebraic [**3**, p. 20].

ADDED IN PROOF. Kevin McCrimmon now has a more direct proof of the theorem.

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