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ON ALGEBRAS SATISFYING THE IDENTITY $(yx)x+x(xy)=2(xy)x$

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ABSTRACT. Simple, strictly power-associative algebras satisfying the identity $(yx)x+x(xy)=2(xy)x$ over a field of characteristic not 2 or 3 have been classified by F. Kosier as commutative Jordan, quasi-associative, or of degree less than three. In the present paper those of degree three or greater are shown to be commutative, which eliminates the quasi-associative case mentioned above.

According to a result of F. Kosier [**2**, Theorem 4.7, p. 317], the simple, strictly power-associative algebras over a field of characteristic not 2 or 3 and satisfying the identity

$$(yx)x + x(xy) = 2(xy)x \tag{1}$$

may be characterized as being either of degree less than three, noncommutative Jordan, or quasi-associative. It will be shown in the following that this list of possibilities can be reduced and the following theorem is proved.

THEOREM. *A simple, strictly power-associative algebra over a field of characteristic not* 2 *or* 3 *and which satisfies* (1) *is*

(a) *a commutative Jordan algebra*;
(b) *an algebra of degree* 2; *or*
(c) *an algebra of degree* 1.

To prove this theorem we will take advantage of the earlier mentioned result due to Kosier and assume in what follows that A is a simple, strictly power-associative algebra of degree exceeding 2 over a field of characteristic not two or three and satisfying (1). By that result, A is then either a Jordan algebra or a quasi-associative algebra and thus in either case is a noncommutative Jordan algebra [**1**, Theorem 2, p. 582]. Since the objective is to show that A is commutative and since A is commutative if and only if every scalar extension is commutative, we may assume that K is an algebraically closed field.

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Notations used here include (x, y, z) to denote $(xy)z-x(yz)$ and $x{\cdot}y$ to denote $xy+yx$. Noncommutative Jordan algebras satisfy the identities

$$(2) \qquad F(x, y, z) = 0 \quad \text{where} \quad F(x, y, z) = (x, y, z) + (z, y, x),$$

$$(3) \qquad J(x, y, z, w) = 0$$

where

$$J(x, y, z, w) = (x, y, z \cdot w) + (z, y, w \cdot x) + (w, y, x \cdot z).$$

The identity (2) is the linearization of the flexible law, $(x, y, x) = 0$.

Advantage will be taken here of well-known ([**1**], [**3**]) facts regarding idempotents in a noncommutative Jordan algebra. Included in these is the vector space direct sum decomposition relative to any idempotent e; $A=A(e, 0)+A(e, 1)+A(e, 2)$ where $A(e, 2)=\{x \text{ in } A: e{\cdot}x=\lambda x\}$ for $\lambda=0$, 1, or 2. Then $A(e, 0)A(e, 2)=A(e, 2)A(e, 0)=0$, the subspaces $A(e, \lambda)$ are subalgebras for $\lambda=0$ or 2, and $A(e, \lambda)A(e, 1)+A(e, 1)A(e, \lambda)\subseteq A(e, 1)$ for $\lambda=0$ or 2. The last property is referred to as stability. Also for $\lambda=0$ or 2 and x in $A(e, \lambda)$, $2ex=2xe=\lambda x$. Since the degree of A exceeds 2 and since K is algebraically closed, there are pairwise orthogonal idempotents e_1, e_2, and e_3 such that $e_1+e_2+e_3=1$. Relative to these three idempotents A has the decomposition, $A=\sum A_{ij}$, $1\leqq i, j\leqq 3$, where $A_{ii}=A(e_i, 2)$ and for $i\neq j$, $A_{ij}=A_{ji}=A(e_i, 1)\cap A(e_j, 1)$. For i, j, and k pairwise distinct, these subspaces have the properties $A_{ii}A_{ij}+A_{ij}A_{ii}\subseteq A_{ij}$, $A_{ii}A_{jj}=A_{jj}A_{kk}=A_{kk}A_{ij}=0$, $A_{ij}A_{jk}\subseteq A_{ik}$, $A(e_i, 1)=A_{ij}+A_{ik}$, and $A(e_i, 0)=A_{jj}+A_{jk}+A_{kk}$.

We shall adopt the notation that for e an idempotent, for $\lambda=0, 1$, and 2, and for S a subset of A, $[S]_\lambda$ shall denote the set of all components in $A(e, \lambda)$ of elements of S. Similarly $[S]_{ij}$ denotes the set of components in A_{ij} for elements in S. Then for subspaces S and T, the commutative product $S\circ T$ is defined as $\sum_{\lambda=0,1,2} [ST+TS]_\lambda$ and $S^{(2)}=S\circ S$. Under this agreement, $S\circ T$ contains ST and TS so that a subspace S is an ideal of A if $A\circ S\subseteq S$.

If e is any idempotent then the subspace $C(e)$ shall denote the set $\{x \text{ in } A(e, 1): 2ex=x\}$. Then $M(e)$ denotes the subspace $C(e)+C(e)\circ A(e, 1)$. The subspace $C(e_1)$ is singled out for special attention and is denoted simply by C. Similarly, M denotes $M(e_1)$. The proof of the theorem stated above proceeds by showing that M is an ideal of A. This fact along with the simplicity of A yields the equality $C=A(e, 1)$. One can move then with reasonable dispatch to the commutativity of A. It is necessary to first deduce some preliminary lemmas.

LEMMA 1. *If, for any idempotent e, x and y are in $A(e, 1)$ and z is in $A(e, \lambda)$ for $\lambda=0$ or 2 then*

$$(4) \qquad [xy]_\lambda z = [x(yz) + (1 - \lambda)(ey - \tfrac{1}{2}\lambda y)(z \cdot x)]_\lambda$$

and

$$(5) \qquad z[yx]_\lambda = [(zy)x + (\lambda - 1)(ey - \tfrac{1}{2}\lambda y)(z \cdot x)]_\lambda.$$

PROOF. Expanding the identity $J(x, y, e, z) - F(z, y, x) = 0$ yields $(xy)z = x(yz) + (1-\lambda)(e, y, z\cdot x)$. Equating the components in $A(e, \lambda)$ and noting that $[e(y(z\cdot x))]_\lambda = [\frac{1}{2}\lambda y(z\cdot x)]_\lambda$ gives the identity (4). The identity (5) is obtained similarly by expanding $J(x, y, e, z) - \lambda F(z, y\, x) = 0$.

LEMMA 2. *If e is any idempotent then the subspace $H(e) = A(e, 1) + A(e, 1)^{(2)}$ is an ideal of A.*

PROOF. Stability and the definition of $H(e)$ yield immediately that $A \circ A(e, 1) \subseteq H(e)$ and that $A(e, 1) \circ A(e, 1)^{(2)} \subseteq H(e)$. Let x, y, and z be as in Lemma 1. Then $z[xy]_\lambda$ and $[xy]_\lambda z$ are in $H(e)$ since the right members of the identities (4) and (5) are in $H(e)$. Thus, for $\lambda = 0$ or 2, $A(e, \lambda) \circ A(e, 1)^{(2)} \subseteq H(e)$ and $H(e)$ is an ideal.

LEMMA 3. *Relative to the idempotents e_1, e_2, and e_3 the equality $A_{ij} = A_{ik}A_{kj} + A_{kj}A_{ik}$ holds.*

PROOF. By the previous lemma, $H(e_i + e_j)$ is an ideal of A. The simplicity of A yields $H(e_i + e_j) = A$. The components, A_{ij} and $[H(e_i + e_j)]_{ij}$, of these spaces are then equal and

$$[H(e_i + e_j)]_{ij} = [(A_{ik} + A_{jk})^2]_{ij} = A_{ik}A_{jk} + A_{jk}A_{ik}.$$

LEMMA 4. *The subspace $[C(e_i)]_{ij}$ is contained in the subspace $C(e_i)$.*

PROOF. If y is in $[C(e_i)]_{ij}$ then $y + z = x$ for some z in $[C(e_i)]_{ik}$ and x in $C(e_i)$. Then $y + z = x = 2e_ix = 2e_iy + 2e_iz$, so since e_iy is in A_{ij} and e_iz is in A_{ik}, it follows that $y = 2e_iy$.

LEMMA 5. *If $i \neq j$ then $[C(e_i)]_{ij} = [C(e_j)]_{ij}$.*

PROOF. Let x be in $[C(e_j)]_{ij}$. Then $xe_i - e_ix = 2F(e_i, e_j, x) = 0$. This implies that x is in $[C(e_i)]_{ij}$. Since i and j are arbitrary this completes the proof.

From this point on, C_{ij} will denote the subspace $[C(e_i)]_{ij} = [C(e_j)]_{ij}$.

LEMMA 6. *If e is an idempotent, y in $A(e, \lambda)$ for $\lambda = 0$ or 2, and x in $C(e)$ then $xy = yx$.*

PROOF. Expanding $2F(y, e, x) = 0$ yields $[1 - \lambda](xy - yx) = 0$ and since $\lambda \neq 1$, $xy = yx$.

LEMMA 7. If *y is in $A(e, \lambda)$ for $\lambda = 0$ or 2 then $yC(e) + C(e)y \subseteq C(e)$.*

PROOF. Let x be in $C(e)$. By Lemma 6 and by identity (2), $F(e, x, y)+(e-\frac{1}{2})(xy-yx)=0$. This expands to $(yx)e=e(yx)$. Since yx is in $A(e, 1)$ then yx is in $C(e)$. That xy is in $C(e)$ then follows from $xy=yx$.

LEMMA 8. *The product $A(e_1, 1)^{(2)}$ is contained in $A(e_1, 0)+A(e_1, 2)$.*

PROOF. Since $A(e_1, 1)=A_{12}+A_{13}$, the desired containment follows if the component in A_{ij} of $(A_{ij})^2$ is zero for $j=2$ and 3. In (4) relative to the idempotent e_k where $k\neq j$ and $k=2$ or 3 let x and y be chosen, one each, from A_{1k} and A_{jk}. Let z be in A_{1j} and let λ be 0. Then the right member of (4) has zero as its component in A_{1j} so $[xy]_0z$ is in $A_{11}+A_{jj}$. But since $A_{1j}=A_{1k}A_{kj}+A_{kj}A_{1k}$ it follows that $(A_{1j})^2\subseteq A_{11}+A_{jj}$ and the desired result is achieved.

LEMMA 9. *The subspace $M=C+A(e_1, 1)\circ C$ is an ideal of A.*

PROOF. It is immediate from Lemma 7 that $A\circ C$ is contained in M. Since $C\circ A(e_1, 1)\subseteq A(e_1, 0)+A(e_1, 2)$ by Lemma 8 and since $A(e_1, 0)\circ A(e_1, 2)=0$ it suffices to show that $A(e_1, \lambda)\circ[C\circ A(e_1, 1)]_\lambda$ is contained in M for $\lambda=0$ and 2 and to show that $A(e_1, 1)\circ[C\circ A(e_1, 1)]_\lambda$ is contained in M. The first containment follows readily from the identities (4) and (5) since if z is in $A(e_1, \lambda)$ and if x and y are selected in any order from C and $A(e_1, 1)$ then the right members are in M. The second containment may be obtained by considering the various subspaces A_{ij} and C_{ij} since $A(e_1, 1)=A_{12}+A_{13}$ and $C=C_{12}+C_{13}$. Since $C_{ij}\subseteq C(e_k, 1)$ and $A_{ij}A_{jk}\subseteq A_{ik}$ for any i, j, and k, $A_{ij}\circ C_{jk}\subseteq C_{ik}$ follows from $A_{ij}\subseteq A(e_k, 0)$. Thus $A_{1j}\circ(A_{1j}\circ C_{1k})+A_{1j}\circ(A_{1k}\circ C_{1j})\subseteq C_{1k}\subseteq M$ for (j, k) equal to (2, 3) or (3, 2). By selecting z from A_{1j} and x and y one each from A_{1k} and C_{1k} in (4) and (5) relative to e_k, it can be shown that $A_{1j}\circ(A_{1k}\circ C_{1k})\subseteq C_{1j}\subseteq M$ for (j, k) equal to (3, 2) or (2, 3). Finally, for (j, k) equal to (3, 2) or (2, 3), $A_{1j}\circ(A_{1j}\circ C_{1j})=(A_{1k}\circ A_{kj})\circ(A_{1j}\circ C_{1j})$ by Lemma 3. Then relative to e_k in (4) and (5) and using the components of $A_{1j}\circ C_{1j}$ as z we have $(A_{1k}\circ A_{kj})\circ(A_{1j}\circ C_{1j})\subseteq A_{1k}\circ[A_{kj}\circ(A_{1j}\circ C_{1j})]+A_{kj}\circ[A_{1k}\circ(A_{1j}\circ C_{1j})]$. Now $A_{kj}\circ(A_{1j}\circ C_{1j})$ is in $M(e_j)$ the proof being analogous to the argument earlier in the proof of this lemma that $A_{1j}\circ(A_{1k}\circ C_{1k})$ is in M. Since $A_{kj}\circ(A_{1j}\circ C_{1j})$ is also contained in $A(e_j, 1)$ it is in C_{kj}. Therefore $A_{1k}\circ[A_{kj}\circ(A_{1j}\circ C_{1j})]\subseteq A_{1k}\circ C_{kj}\subseteq A_{1j}\subseteq M$. A similar argument yields $A_{kj}\circ[A_{1k}\circ(A_{1j}\circ C_{1j})]$ in M. This completes the proof of the lemma.

In a simple, power-associative, flexible algebra with orthogonal idempotents e and f the subspace $A(e, 1)$ is not the zero subspace since otherwise A is the direct product of the ideals $A(e, 0)$ and $A(e, 2)$. For any element x in $A(e, 1)$, $ex=xe$ if and only if $2ex=x$. Let y be a nonzero element of $A(e_1, 1)$. Then letting x in (1) be e_1 gives $(ye_1)e_1+e_1(e_1y)=2(e_1y)e_1$. By the flexibility of A, $(e_1y)e_1=e_1(ye_1)$ so $(ye_1)e_1-(e_1y)e_1=e_1(ye_1)-e_1(e_1y)$. Thus

$(ye_1-e_1y)e_1=e_1(ye_1-e_1y)$ and hence depending on whether ye_1-e_1y is zero or not zero, either y or ye_1-e_1y is a nonzero element of C. Since $M\supseteq C$ and A is simple, $M=A$. This implies that $A(e_1, 1)=C$.

LEMMA 10. *If x and y are in $A(e_1, 1)$ then $xy=yx$.*

PROOF. By the above, x and y are in C. Expanding $2[F(e_1, x, y)]_\lambda=0$ for $\lambda=0$ or 2 gives $(1-\lambda)[xy-yx]_\lambda=0$ so $[xy]_\lambda=[yx]_\lambda$. Thus, since by Lemma 8, xy and yx are in $A(e_1, 0)A+(e_1, 2)$, $xy=[xy]_0+[xy]_2=[yx]_0+[yx]_2=yx$ proving the lemma.

By Lemmas 6 and 10, $xy=yx$ for each x in A and y in $A(e_1, 1)$. Thus to show that A is commutative it is only necessary to show that $A(e_1, \lambda)$ is a commutative subalgebra for $\lambda=0$ and 2. This is the substance of the final lemma.

LEMMA 11. *If x and y are in $A(e_1, \lambda)$ for $\lambda=0$ or 2 then $xy=yx$.*

PROOF. By Lemma 2, $H(e_1)$ is an ideal of A and by the simplicity of A, $H(e_1)=A$. Thus $A(e_1, \lambda)\subseteq A(e_1, 1)^{(2)}$. Then the desired result follows if $x(zw)=(zw)x$ for z and w in $A(e_1, 1)$ and x in $A(e_1, \lambda)$. But $F(z, w, x)=0$ and by Lemmas 6 and 10, $z(wx)=(xw)z$ so $(zw)x-x(zw)=(zw)x-x(wz)=F(z, w, x)+z(wx)-(xw)z=0$. This shows that $A(e_1, \lambda)$ is commutative.

Lemma 11 completes the argument that A is commutative and proves the above theorem.

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