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ON THE COUNTABLY GENERATED z-IDEALS OF $C(X)$

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ABSTRACT. A necessary and sufficient condition for the countable generation of certain z-ideals of $C(X)$ is given. In particular, for X compact, the countably generated z-ideals of $C(X)$ are the sets of all functions which vanish on a neighborhood of some zero-set of X. Any finitely generated semiprime ideal of $C(X)$ is generated by an idempotent.

1. **Introduction.** In this paper we establish a necessary and sufficient condition for certain z-ideals of $C(X)$ to be countably generated. An immediate corollary of this result is: if X is compact T_2, then the countably generated z-ideals of $C(X)$ are precisely the ideals $O_A=\bigcap_{p\in A} O_p$, where A is a zero-set of X. The proof (§2) uses an improvement of an argument first introduced in [G], and successively modified in [K]. In §3 some complementary results are obtained. For terminology and notations the reader is referred to [GJ], [G] and [K].

2. **Countably generated z-ideals.**

LEMMA 2.1. *Let A be closed in βX. The ideal $O^A=\bigcap_{p\in A} O^p$ is countably generated if and only if A is a zero-set of βX.*

PROOF. If $O^A=(f_1,f_2,\cdots)$ then $A=\bigcap_n \operatorname{int}_{\beta X}\operatorname{cl}_{\beta X}Z(f_n)$ is a closed G_δ in the compact space βX and hence a zero-set. Conversely, assume $A=Z_{\beta X}(g)$, where $g\in C(\beta X)$, $g\geqq 0$. Put $g_n=(g-1/n)\vee 0$ for $n=1,2,\cdots$. A standard compactness argument shows that the sets $Z_{\beta X}(g_n)$ are a neighborhood base for A in βX; hence, by [GJ, 7.0.2], $O^A=(g_1|X, g_2|X,\cdots)$.

LEMMA 2.2. *Let I be a countably generated ideal of $C(X)$, and let A be a zero-set of βX contained in $\theta(I)=\{p\in\beta X: M^p\supseteq I\}$. Then every z-ideal J of $C(X)$ contained in I is contained in $O^A=\bigcap_{p\in A} O^p$.*

PROOF. The proof closely resembles that of [K, Lemma 1], except that some more technicalities are required here.

Assume that $J\nsubseteq O^A$; then there exists $Z\in \boldsymbol{Z}[J]$ such that $\operatorname{int}_{\beta X}\operatorname{cl}_{\beta X} Z\nsupseteq A$; we will show that there exists $h\in C(X)$ such that $Z(h)\supseteq Z$, but $h\notin I$. This

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implies $h\in J\setminus I$, contrary to the assumption $J\subseteq I$. Let $g\in C(\beta X)$ be such that $Z_{\beta X}(g)=A$. Since g is continuous and $\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z\not\supseteq A$, we have $0\in \mathrm{cl}_{\mathbf{R}}(g[X\setminus Z])$. Thus there exists a countable discrete subset of $X\setminus Z$, say $D=\{x_1, x_2, \cdots\}$, such that $\lim_n g(x_n)=0$. Hence $\mathrm{cl}_{\beta X}(D)\setminus D\subseteq Z_{\beta X}(g)=A\subseteq \mathrm{cl}_{\beta X}Z$ ($J\subseteq I$, by hypothesis), so that $S=D\cup \mathrm{cl}_{\beta X}Z$ is closed in βX. Assume now that $I=(f_1, f_2, \cdots)$ where $|f_i|\leqq 1$ for all $i=1, 2, \cdots$ and put $s=\sum_n 2^{-n}|f_n|^{1/2}$. Since $\lim_n s(x_n)=0$, and $\mathrm{cl}_{\beta X}(D)\setminus D\subseteq \mathrm{cl}_{\beta X}Z$, the function h_0 defined on S by means of $h_0|D=s|D$, $h_0|\mathrm{cl}_{\beta X}Z=0$, is continuous on the compact set S. Take $h_1\in C(\beta X)$ such that $h_1|S=h_0$, and put $h=h_1|X$. Then $h\in C(X)$, and $Z(h)\supseteq Z$. Suppose that $h=\sum_{i=1}^k h_i f_i$, with $h_i\in C(X)$. For $x_n\in D$ we have

$$h(x_n) = s(x_n) = \sum_{i=1}^{k} h_i(x_n) f_i(x_n),$$

hence

$$1 \leqq \sum_{i=1}^{k} |h_i(x_n)|\, |f_i(x_n)|^{1/2}\, 2^i.$$

Let u^* be the Stone extension of the function $u=\sum_{i=1}^k |h_i|\,|f_i|^{1/2}\,2^i$ (see [**GJ**, 7.5]). For each $p\in\theta(I)$, $u\in M^p$, i.e., $u^*(p)=0$. But $u(x_n)\geqq 1$ for all $x_n\in D$, hence $u^*(p)\neq 0$ for all $p\in \mathrm{cl}_{\beta X}(D)\setminus D\subseteq \theta(I)$, a contradiction.

THEOREM. *Let I be a z-ideal of $C(X)$, and suppose that $\theta(I)$ is a zero-set of βX. Then I is countably generated if and only if $I=\bigcap_{p\in\theta(I)} O^p$.*

PROOF. By Lemma 2.1, $O^{\theta(I)}=\bigcap_{p\in\theta(I)} O^p$ is countably generated. By Lemma 2.2, if I is countably generated, then $O^{\theta(I)}$ is the largest z-ideal of $C(X)$ contained in I.

COROLLARY. *If X is a compact Hausdorff space, then the countably generated z-ideals of $C(X)$ are the ideals $\bigcap_{p\in A} O_p$, where A is a zero-set of X.*

REMARK. [**K**, Theorem 1] is easily obtained from the preceding Lemma 2.2 (if $p\in X$ has a countable base of neighborhoods in X, then $\{p\}$ is a zero-set of βX).

Question. Is every countably generated z-ideal of $C(X)$ of the form $\bigcap_{p\in A} O^p$, with A a zero-set of βX?

3. Miscellaneous results. Recall that an ideal I is said to be *semiprime* if $f^2\in I$ implies $f\in I$.

THEOREM. (a) *Every principal semiprime ideal I of $C(X)$ is generated by an idempotent.*

(b) *Every finitely generated semiprime ideal and every countably generated intersection of real maximal ideals are principal, and hence generated by an idempotent.*

PROOF. (a) Let $I=(f)$; since $f^{1/3}\in I$, we have $f^{1/3}=hf$, with $h\in C(X)$. Thus $g=h(f^2)^{1/3}$ is an idempotent. Clearly, $(g)=I$.

(b) Suppose that $I=(f_1, f_2, \cdots)$ where $|f_i|\leqq 1$ for all $i=1, 2, \cdots$ (in the first case, we assume also $f_i=0$ for i sufficiently large), and put $s=\sum_n 2^{-n}|f_n|^{1/2}$. In the first case, $s\in I$ since I is semiprime and the sum is finite. In the second case, $s\in I$ since $\boldsymbol{Z}(s)=\bigcap_n \boldsymbol{Z}(f_n)$, so that $s\in M$ for every real maximal M containing I. Define now h_i on X by means of $h_i(x)=f_i(x)/s(x)$ for $x\notin \boldsymbol{Z}(s)$, $h_i(x)=0$ for $x\in \boldsymbol{Z}(s)$. Since $|h_i|\leqq |f_i|^{1/2}\, 2^i$, h_i is continuous; thus $f_i=h_i s\in(s)$, hence $I\subseteq (s)$; but $s\in I$, so that $(s)=I$.

REMARK 1. The first case of (b) generalizes **[G,** 6.6] and **[D,** 2.6]. The second improves **[G,** 5.2] (except that we assume here $J=C(X)$).

REMARK 2. An intersection of hyper-real maximal ideals may be countably generated without being trivial. Let A be a zero-set of βX contained in $\beta X\backslash X$; by Lemma 2.1, $\boldsymbol{O}^A=\bigcap_{p\in A}\boldsymbol{O}^p$ is countably generated; and by **[M,** Theorem 5.3], $\boldsymbol{O}^A=\bigcap_{p\in A}\boldsymbol{M}^p$. Thus, unless X is pseudocompact, there exists a closed nonopen $A\subset\beta X$ such that $\bigcap_{p\in A}\boldsymbol{M}^p$ is a countably generated (nontrivial) ideal of $C(X)$. This disproves a conjecture raised in **[D,** p. 69].

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