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ON SPANIER'S HIGHER ORDER OPERATIONS

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ABSTRACT. Given a stack A over a simplicial set K, we construct the generalized Eilenberg-Mac Lane complexes $K(A, n)$ that represents a stack cohomology theory over K. We show that the higher order operations defined by Spanier are some sections of $K(A, n)$ over K for some stack A.

Introduction. Spanier in [2] defined higher order operations as obstructions to extending null-homotopies of maps carried by carriers over an abstract simplicial complex and showed that these operations are subsets of cohomology groups of the complex with coefficients in a "stack." In this note we shall construct a stack cohomology theory over a simplicial set (semisimplicial complex) K and prove that it is representable by generalized Eilenberg-Mac Lane complexes $K(A, n)$ constructed in §3. The higher order operations are then homotopy classes of cross sections of $K(A, n)$ over K. An example is given to show that when the stack A is appropriately chosen, various cohomology operations can be represented by cross sections of $K(A, 1)$ over I, the standard 1-simplex Δ^1.

1. **Injective stacks.** As in [1] a simplicial set X is regarded as a category of simplexes with morphisms defined by face operators and degeneracy operators. For example, if $d^i x$ denotes the ith face of the n-simplex $x \in X_n$, then there is a morphism $d^i_x : x \to d^i x$. A *prestack* over X with values in a category $\mathscr{A}$ is by definition a contravariant functor $A : X \to \mathscr{A}$; A is called a *stack* if, for every morphism $s^i_x : x \to s^i x$ defined by the ith degeneracy operators s^i, $A(s^i_x) : A(s^i x) \to A(x)$ is an isomorphism in $\mathscr{A}$. The category of prestacks over X with values abelian groups is then a functor category; it is denoted by $\mathscr{A}b_X$.

Let $f : X \to Y$ be a simplicial map. Then f is a (covariant) functor when the simplicial sets X and Y are regarded as categories. f induces two functors $f^{\#} : \mathscr{A}b_Y \to \mathscr{A}b_X$ and $f_{\#} : \mathscr{A}b_X \to \mathscr{A}b_Y$, where $f^{\#}B = Bf$ and $(f_{\#}A)(y) = \prod_x A(x)$, $x \in f^{-1}(y)$. $f^{\#}$ is (left) adjoint to $f_{\#}$ and both functors are exact. Consequently, $f_{\#}$ (resp. $f^{\#}$) preserves injectives (resp. projectives).

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Let Δ^n be the standard n-simplex (a simplicial set) with its only nondegenerate n-simplex denoted by δ^n and let $x:\Delta^n \to X$ be the simplicial map defined by the correspondence $\delta^n \to x$, $x \in X_n$. Then the constant stack $Q_{(n)}$ over Δ^n with value the group of rationals mod 1 is an injective stack and therefore $x_\# Q_{(n)}$ is injective in $\mathscr{A}b_X$. Let $Q = \prod_{x \in X} x_\# Q_{(n)}$. Then Q is an injective cogenerator of $\mathscr{A}b_X$ and therefore $\mathscr{A}b_X$ has enough injectives.

2. **Stack cohomology.** A prestack A over X is realized as a family of abelian groups $\{A(x); x \in X\}$ together with two families of homomorphisms

$$A(d^i): A(d^i x) \to A(x) \quad \text{and} \quad A(s^i): A(s^i x) \to A(x).$$

The *cochain complex* of A is $CA = \{C^n A = \prod_x A(x); x \in X_n\}$ with coboundary maps alternating sums of $A(d^i)$. The cohomology groups of CA are denoted $H^*(A)$ or $H^*(X; A)$. Every simplicial map $f: X \to Y$ induces homomorphisms $f^*: H^*(Y; B) \to H^*(X; f^\# B)$, $B \in \mathscr{A}b_Y$, defined by $f^*([c]) = [cf]$. Since the 0th cohomology functor H^0 is continuous and since $H^q(X; Q) = 0$ for $q > 0$ and Q injective, H^* are derived functors of H^0.

Let K be a fixed simplicial set. Dual to the homology theory in **[1]** we have a cohomology theory over K which can be computed by injective resolutions and which is unique in the sense of Eilenberg-Steenrod. For our use in this note we shall paraphrase only the definition and the homotopy axiom of the theory (in absolute case).

Let $\mathscr{C}_K$ be the category of simplicial sets over K, objects X_φ are simplicial maps $\varphi: X \to K$; morphisms $f: X_\varphi \to Y_\Psi$ are simplicial maps $f: X \to Y$ such that $\Psi f = \varphi$. Given a stack A over K, the *cohomology groups of X_φ with coefficients in A* are $H^*(X_\varphi; A) = H^*(\varphi^\# A)$ as defined in the preceding paragraph. Thus for every map over K, $f: X_\varphi \to Y_\Psi$, f induces homomorphisms

$$f^*: H^*(Y_\Psi; A) \to H^*(X_\varphi; A)$$

which are $f^*: H^*(\Psi^\# A) \to H^*(f^\# \Psi^\# A) = H^*(\varphi^\# A)$.

For each q-simplex $\sigma \in K_q$, let Δ^σ be the simplicial subset of K generated by σ and let i_σ be the inclusion map. A *homotopy over K* is a map F: $(X \times I)_{\varphi p} \to Y_\Psi$ in $\mathscr{C}_K$, where $p(x, \delta) = x$. It is a family of simplicial homotopies $F = \{F_\sigma: (i_\sigma^\# X) \times I \to i_\sigma^\# Y; \sigma \in K\}$.

3. **The representation theorem.** For $\tau = d\sigma$ a face of $\sigma \in K_q$, there is a map $j: \Delta_\tau^{q-1} \to \Delta_\sigma^q$ in $\mathscr{C}_K$ defined by $j(\delta^{q-1}) = d\delta^q$ as in the commutative diagram

$$\begin{tikzcd}
\Delta^{q-1} \arrow[rr, "j"] \arrow[dr, "\tau"'] & & \Delta^q \arrow[dl, "\sigma"] \\
& K &
\end{tikzcd}$$

j induces a homomorphism of groups of normalized n-cocycles

$$Z^n(j):Z^n(\Delta^q;\sigma^\# A)\to Z^n(\Delta^{q-1};j^\#\sigma^\# A)=Z^n(\Delta^{q-1};\tau^\# A).$$

Similarly, for a degeneracy operator s and $\sigma=s\tau$, one defines a homomorphism $Z^n(h)$. As a generalization of Eilenberg-Mac Lane complexes, we define simplicial sets $K(A,n)=\{K_q(A,n);q=0,1,2,\cdots\}$ with $K_q(A,n)=\bigcup_\sigma Z^n(\Delta^q;\sigma^\# A)$, $\sigma\in K_q$, and face operators and degeneracy operators defined by $Z^n(j)$ and $Z^n(h)$ respectively. An *Eilenberg-Mac Lane object* $K(A,n)_\theta$ is then a simplicial map $\theta:K(A,n)\to K$ that sends all cocycles in $Z^n(\Delta^q;\sigma^\# A)$ onto σ. In particular, if $K=\Delta^0$ is a point and if A is the constant stack over K with value group π, then $K(A,n)$ is the Eilenberg-Mac Lane complex $K(\pi,n)$. Notice also that $K(A,n)_\theta$ can be viewed as a functor (covariant!) $E:K\to\mathscr{A}b$ with $E(\sigma)=Z^n(\Delta^q;\sigma^\# A)$.

An element $[c]$ in $H^n(K(A,n)_\theta;A)=H^n(\theta^\# A)$ is said to be *characteristic for* $K(A,n)$ if it is represented by the *fundamental cocycle* c defined by $c(e)=e(\delta^n)$ for every $e\in K_n(A,n)$. Let $[X_\varphi,K(A,n)_\theta]$ be the group of homotopy classes of maps over K from X to $K(A,n)$. Then

THEOREM A. *$[X_\varphi,K(A,n)_\theta]$ is naturally isomorphic to $H^n(X_\varphi;A)$.*

Indeed, $[f]\to f^*([c])$, where c is a fundamental cocycle, defines an isomorphism $c_n:[X_\varphi,K(A,n)_\theta]\to H^n(X_\varphi;A)$ with $c_n^{-1}([h])=[f]$ in $[X_\varphi,K(A,n)]$ defined by the equality $(f(x))(\delta^n)=h(x)$.

In particular, when $K=\Delta^0$ is a point, our results show that the functors $[-,K(\pi,n)]$ are derived functors of $H^0(-;\pi)$.

4. **Higher order operations.** In this section we shall paraphrase the main result and some examples in **[2]** in terms of cross sections of $K(A,n)_\theta$ over K.

Let $\mathscr{A}$ be the category of pointed topological spaces and base point preserving continuous maps and let $T:K\to\mathscr{A}$, $T(\sigma)=|\sigma|$, be the stack of geometrical realizations. For a *carrier* $\Phi:K\to\mathscr{A}$ (a stack!), a stack map $f:T\to\Phi$ is said to be *carried by* Φ. A homotopy $F:f\simeq g$ of the maps f and g carried by Φ is a stack map

$$F=\{F_\sigma:T(\sigma)xI\to\Phi(\sigma);F_\sigma(x,0)=f(x),F_\sigma(x,1)=g(x)\}.$$

Let Γ_n be the stack of groups defined by $\Gamma_n(\sigma)=\pi_n(\Phi(\sigma))$. Then the $(n+1)$-operation $O^n(f)$ is the set of obstructions $c_n(F)$ to extending null-homotopies F on the $(n-1)$-skeleton K^{n-1} of K and so is a subset of $H^n(K;\Gamma_n)$. Let X_φ in Theorem A be $1:K\to K$. Then

$$H^n(K;\Gamma_n)\approx[K,K(\Gamma_n,n)_\theta]\quad\text{for } n\geqq 2$$

and

THEOREM B. *The $(n+1)$-operation $O^n(f)$ (if it exists) is a subset of homotopy classes of cross sections of $K(\Gamma_n, n)_\theta$.*

When $n=1$, various cohomology operations can be represented by cross sections of $K(A, 1)$ over I when A is appropriately chosen. For example, a cross section representation of the Massey triple product is obtained by defining A:

$$H^{p+q-1}(X; G_{12}) \xrightarrow{d} H^{p+q+r-1}(X; G_{123}) \xleftarrow{d'} H^{q+r-1}(X; G_{23})$$

with $d(u)=u\cup u_3$, $d'(v)=u_1\cup v$, $u_1\cup u_2=0=u_2\cup u_3$, and the G's suitably paired.

REFERENCES

1. Y. C. Chen, *Stacks, costacks and axiomatic homology*, Trans. Amer. Math. Soc. **145** (1969), 105–116. MR **40** #3536.

2. E. H. Spanier, *Higher order operations*, Trans. Amer. Math. Soc. **109** (1963), 509–539. MR **28** #1622.

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