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ON GROUPS OF EXPONENT FOUR. II

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ABSTRACT. C. R. B. Wright has shown that the nilpotency class of an n-generator group of exponent four is at most $3n-1$. This bound is believed to be too large for higher values of n. In this note it is shown that if for some large enough integer n this bound can be improved to $[2\frac{1}{2}n]-1$, then the free group of exponent four of infinite rank is solvable.

Introduction. Let $\kappa(n)$ denote the nilpotency class of $F_n(\mathfrak{B}_4)$. Wright [3] has shown that $\kappa(n) \leqq 3n-1$. For higher values of n this bound is believed to be too large. The purpose of this note is to point out a connection with the solvability problem of $F_\infty(\mathfrak{B}_4)$ studied in Gupta-Weston [**2**]. Our main result may be stated as follows:

THEOREM. *Let N be a fixed positive integer such that for all $n \geqq N$, $\kappa(n) \leqq [2\frac{1}{2}n]-1$, then $F_\infty(\mathfrak{B}_4)$ is solvable of bounded length.*

NOTATION. We use standard notation: $[x, y]=x^{-1}y^{-1}xy$; $[x, y, z]=[[x, y], z]$; $[x, y; u, v]=[[x, y], [u, v]]$; G^4 is the subgroup of G generated by 4th powers of elements of G; $W(G)$ is the verbal closure of a word w in G; $\gamma_n(G)$ is the nth term of the lower central series of G; $\delta_m(G)$ is the mth term of the derived series of G; $\langle x^G\rangle$ is the normal closure of x in G; $\mathfrak{B}_4$ is the variety of groups of exponent 4; $F_n(\mathfrak{B}_4)$ is the free $\mathfrak{B}_4$-group of rank n; $F=F_\infty$ is the free group of infinite rank.

Preliminaries. Any r variable commutator word of weight s is said to be of type $(r \to s)$. Consider

$$w_0 = [z_0, y_0, z_0, x_0, x_0, y_0, y_0]$$

to be a word of type $(3 \to 7)$. Using w_0 as a base we shall define words w_n of type $(2\cdot 2^{n+1}-1 \to 5\cdot 2^{n+1}-3)$ inductively. Let $k \geqq 0$ and assume we have already defined w_k of type $(2\cdot 2^{k+1}-1 \to 5\cdot 2^{k+1}-3)$. Let w_{k1}, w_{k2} be two copies of w_k involving nonoverlapping variables; and define

$$w_{k+1} = [w_{k1}, y_{k+1}, w_{k2}, y_{k+1}, y_{k+1}]$$

which is clearly of type $(2\cdot 2^{k+2}-1 \to 5\cdot 2^{k+2}-3)$.

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For our later use we record the following:

LEMMA 1. *Modulo* F^4, (i) $\langle w_n^F\rangle$ *is abelian*; (ii) $w_n^2=1$ *for all* $n\geqq 0$.

PROOF OF (i). The derived group of $\langle w_n^F\rangle$ is the normal closure of all elements of the form $[w_n, x, w_n]$. Since the weight of this commutator is more than three times the number of variables involved, by the result of Wright [3], $[w_n, x, w_n]=1$.

PROOF OF (ii). By the Corollary on p. 220 of Gupta-Tobin [1], w_0^2 is a product of commutators of weight at least 9 in 3 variables, each of which is trivial by the result of Wright. Thus $w_0^2=1$ and together with (i), $\langle w_0^F\rangle$ is of exponent 2. Since, for each $n\geqq 0$, $w_n\in\langle w_0^F(n)\rangle$ for some copy $w_0(n)$ of w_0, it follows that $w_n^2=1$. This completes the proof of Lemma 1.

Let H be a group of exponent 4 generated by $x_1, x_2, \cdots$ and satisfying only the following relations and their consequences:

(I) $x_i^2=1$,

(II) $[x_i, h, x_i]=1$ for all $i=1, 2, \cdots$ and all $h\in H$.

LEMMA 2 (GUPTA-WESTON [2]). (i) *H satisfies* $[x_i, h, h, h]=1$ *for all* $i=1, 2, \cdots$ *and all* $h\in H$.

(ii) $\delta_k(H)=\{1\}$ *implies* $\delta_{k+2}(F)<F^4$.

(iii) *If H satisfies the relations* $[x_i, h, x_j, h, h]$ *for all* $i, j\in\{1, 2, \cdots\}$ *and all* $h\in H$, *then H is centre-by-metabelian*. ((iii) *follows from the proof of Theorem* 2 of [2].)

LEMMA 3.[1] *Let h be an arbitrary but fixed element of H and let K_n $(n\geqq 1)$ denote the subgroup of H generated by $x_1, \cdots, x_n$, h. Then* $\gamma_{n+4}(K_n)=\{1\}$.

PROOF. We may assume that $\gamma_{n+5}(K_n)=\{1\}$. Since $\langle x_i^H\rangle$ is abelian for each i, $\gamma_{n+4}(K_n)$ is generated by commutators of weight $n+4$ each of which contains at least four h's. Thus for $n\geqq 2$, $\gamma_{n+4}(K_n)=\{1\}$ by Theorem 1 of Wright [3] and for $n=1$, $\gamma_5(K_1)=\{1\}$ by Lemma 2(i).

LEMMA 4. $\delta_4(H)\leqq W_0(H)$.

PROOF. Assume $W_0(H)=\{1\}$, so that H satisfies $w_0=1$. In particular for all $h\in H$,

$$\begin{aligned}
1 &= [x_1x_2, h, x_1x_2, x_3x_4, x_3x_4, h, h]\\
&= [x_1, h, x_2, x_3x_4, x_3x_4, h, h][x_2, h, x_1, x_3x_4, x_3x_4, h, h]\\
&= [[x_1, x_2], h, x_3x_4, x_3x_4, h, h] \quad \text{(by Jacobi identity and Lemma 3)}\\
&= [[x_1, x_2], h, x_3, x_4, h, h][[x_1, x_2], h, x_4, x_3, h, h]\\
&= [[x_1, x_2], h, [x_3, x_4], h, h] \quad \text{(by Jacobi identity and Lemma 3).}
\end{aligned}$$

[1] The authors wish to thank the referee for suggesting Lemma 3 which has simplified the proof of Lemma 4.

By Lemma 2(iii), $\delta_1(H)$ is centre-by-metabelian and in particular $\delta_4(H) \leqq W_0(H)$.

LEMMA 5. *$\delta_{3n+4}(H) \leqq W_n(H)$ for all $n \geqq 0$.*

PROOF. By induction on n. For $n=0$, the result comes from Lemma 4. For the inductive step we assume $\delta_{3k+4}(H) \leqq W_k(H)$ and proceed to conclude that $\delta_{3k+7}(H) \leqq W_{k+1}(H)$. We may assume $W_{k+1}(H)=\{1\}$ so that H satisfies the law $w_{k+1}=[w_{k1}, h, w_{k2}, h, h]$. Let $\bar{H}$ denote the subgroup of H generated by $w_{k1}, w_{k2}, \cdots, w_{k5}$ where each w_{ki} is a copy of w_k. By Lemma 1, $\bar{H}$ is a homomorphic image of H and, by Lemma 2, $\bar{H}$ is centre-by-metabelian. Thus $\delta_3(W_k(H)) \leqq W_{k+1}(H)$ and by induction hypothesis $\delta_{3k+7}(H)=\delta_3(\delta_{3k+4}(H)) \leqq \delta_3(W_k(H)) \leqq W_{k+1}(H)$. This concludes the proof of Lemma 5.

PROOF OF THE THEOREM. Choose $n \geqq N$ of the form $2 \cdot 2^{l+1}-1$. Since $\kappa(n) \leqq [2\frac{1}{2}n]-1$, every commutator of weight $[2\frac{1}{2}n]$ in n variables is trivial mod F^4. In particular every commutator of type $(2 \cdot 2^{l+1}-1 \rightarrow 5 \cdot 2^{l+1}-3)$ is trivial mod F^4 and in turn $W_l(H)=\{1\}$. Thus by Lemma 5, $\delta_{3l+4}(H)=\{1\}$. Now by Lemma 2(ii), $\delta_{3l+6}(F_\infty(\mathfrak{B}_4))=\{1\}$, as was required.

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