PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 2, February 1972

REPRESENTATIONS OF EULER CLASSES

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To Professor S. S. Chern on his sixtieth birthday

ABSTRACT. For any endomorphism K of an oriented module F with inner product there is an element $\mathrm{pf}\, K$ in the ground ring R, a constant multiple of the classical pfaffian in the case $F=R^{2n}$. If R is the algebra of even-dimensional differential forms on a smooth manifold, and if F is the tensor product of R and the module of sections of an oriented $2n$-plane bundle, then any connection in the bundle induces a curvature transformation $K:F\to F$ for which $(4\pi)^{-n}\,\mathrm{pf}\, K$ represents the Euler class. Properties of Euler classes are immediate consequences of corresponding properties of pf.

1. **Pfaffians.** Let R be any commutative ring with unit. An R-module F is *orientable* if $\bigwedge^m F$ is free of rank 1 for some $m>0$; according to **[4]** there is at most one such integer m, and throughout this note we assume that m is an even integer $2n$. An orientable module F is *oriented* by assigning a generator $S\in\bigwedge^{2n} F$.

An *inner product* on F is any nondegenerate symmetric bilinear map $\langle\ ,\ \rangle: F\times F\to R$, not necessarily positive-definite in any sense. An inner product on F induces inner products on $\bigwedge^p F$ for $p=0,\cdots,2n$ in the usual fashion, and we assume $\langle S, S\rangle=\pm 1$ for the orientation S; the $\pm$ sign depends on $\langle\ ,\ \rangle$. The *Hodge operator* $*:\bigwedge^n F\to\bigwedge^n F$ in dimension n is given by requiring

$$\langle e_1\wedge\cdots\wedge e_n, *(f_1\wedge\cdots\wedge f_n)\rangle = \langle e_1\wedge\cdots\wedge e_n\wedge f_1\wedge\cdots\wedge f_n, S\rangle$$

for any $e_1,\cdots,e_n,f_1,\cdots,f_n\in F$.

DEFINITION 1.1. For any endomorphism K of F the *pfaffian* is the element $\mathrm{pf}\, K=(-1)^{n(n-1)/2}\,\mathrm{tr}\, *\circ\bigwedge^n K$, where $*\circ\bigwedge^n K$ is a composition of endomorphisms of $\bigwedge^n F$ and "tr" means "trace".

Let ${}^tK$ denote the transpose of K: $\langle e, {}^tKf\rangle=\langle Ke, f\rangle$.

PROPOSITION 1.2. $\mathrm{pf}\, K=\mathrm{pf}(-{}^tK)$.

Received by the editors December 14, 1970 and, in revised form, March 19, 1971.
AMS 1970 *subject classifications*. Primary 15A15, 57D20, 58A10; Secondary 53B20, 58A99.
Key words and phrases. Pfaffians, Hodge operator, connections, curvature transformations, Euler classes, Avez-Chern theorem, Gauss-Bonnet theorem.
[1] Research supported by National Science Foundation Grant GP-13145.

PROOF. One easily computes ${}^t{*}=(-1)^n{*}$, and since ${}^t(\bigwedge^n K)=\bigwedge^n({}^tK)$ it follows that $\operatorname{tr} {*}\circ\bigwedge^n K=\operatorname{tr}\bigwedge^n K\circ{*}=\operatorname{tr}{}^t(\bigwedge^n K\circ{*})=\operatorname{tr}{}^t{*}\circ{}^t(\bigwedge^n K)=\operatorname{tr}{*}\circ\bigwedge^n(-{}^tK)$.

We shall need the following two properties of $*$.

LEMMA 1.3. *$\bigwedge^n {}^tA\circ{*}\circ\bigwedge^n A=(\det A){*}$ for any $A\in\operatorname{Hom}(F,F)$.*

PROOF.

$$\begin{aligned}\langle e_1\wedge\cdots\wedge e_n, (\textstyle\bigwedge^n {}^tA\circ{*}\circ\bigwedge^n A)f_1\wedge\cdots\wedge f_n\rangle&\\ &=\langle Ae_1\wedge\cdots\wedge Ae_n, {*}(Af_1\wedge\cdots\wedge Af_n)\rangle\\ &=\langle Ae_1\wedge\cdots\wedge Ae_n\wedge Af_1\wedge\cdots\wedge Af_n, S\rangle\\ &=(\det A)\langle e_1\wedge\cdots\wedge e_n, {*}(f_1\wedge\cdots\wedge f_n)\rangle.\end{aligned}$$

LEMMA 1.4. *For any invertible $A\in\operatorname{Hom}(F,F)$ let $\langle\ ,\ \rangle'$ be the inner product given by $\langle e,f\rangle'=\langle Ae,Af\rangle$; then the corresponding Hodge operator ${*}':\bigwedge^n F\to\bigwedge^n F$ satisfies ${*}'=\bigwedge^n A^{-1}\circ{*}\circ\bigwedge^n A$.*

PROOF. For $S'=(\bigwedge^{2n} A^{-1})S$ one has

$$\langle S', S'\rangle'=\langle(\textstyle\bigwedge^{2n} A)S', (\bigwedge^{2n} A)S'\rangle=\langle S,S\rangle=\pm1;$$

hence

$$\begin{aligned}\langle e_1\wedge\cdots\wedge e_n, {*}'(f_1\wedge\cdots\wedge f_n)\rangle'&\\ &=\langle e_1\wedge\cdots\wedge e_n\wedge f_1\wedge\cdots\wedge f_n, S'\rangle'\\ &=\langle Ae_1\wedge\cdots\wedge Ae_n\wedge Af_1\wedge\cdots\wedge Af_n, S\rangle\\ &=\langle Ae_1\wedge\cdots\wedge Ae_n, {*}(Af_1\wedge\cdots\wedge Af_n)\rangle\\ &=\langle e_1\wedge\cdots\wedge e_n, (\textstyle\bigwedge^n A^{-1}\circ{*}\circ\bigwedge^n A)f_1\wedge\cdots\wedge f_n\rangle'.\end{aligned}$$

PROPOSITION 1.5. $\operatorname{pf}(A\circ K\circ{}^tA)=(\det A)\operatorname{pf} K$.

PROOF. Lemma 1.3 gives

$$\begin{aligned}\operatorname{tr}({*}\circ\textstyle\bigwedge^n(A\circ K\circ{}^tA))&=\operatorname{tr}({*}\circ\textstyle\bigwedge^n A\circ\bigwedge^n K\circ\bigwedge^n{}^tA)\\ &=\operatorname{tr}(\textstyle\bigwedge^n{}^tA\circ{*}\circ\bigwedge^n A\circ\bigwedge^n K)\\ &=(\det A)\operatorname{tr}({*}\circ\textstyle\bigwedge^n K).\end{aligned}$$

COROLLARY 1.6. *If $K=A\circ J\circ{}^tA$ then* $(\operatorname{pf} K)^2(\det J)=(\operatorname{pf} J)^2(\det K)$.

PROOF. Both sides are equal to $(\det A)^2(\operatorname{pf} J)^2(\det J)$.

If E and F are oriented inner product spaces of dimensions $2p$ and $2q$ then the direct sum $E\oplus F$ is also an oriented inner product space with respect to the obvious definitions. Let ${*}''$ be the resulting Hodge operator on $\bigwedge^{p+q}(E\oplus F)$, this module being canonically isomorphic to the direct sum $\sum_{r+s=p+q}\bigwedge^r E\otimes\bigwedge^s F$. If ι and ρ are the injection and projection for the summand $\bigwedge^p E \otimes \bigwedge^q F$ then for the Hodge operators $*$ and $*'$ on $\bigwedge^p E$ and $\bigwedge^q F$ one easily verifies $\rho \circ *'' \circ \iota = (-1)^{pq} * \otimes *'$.

PROPOSITION 1.7. *For any $J: E \to E$ and $K: F \to F$ it follows that* $\operatorname{pf}(J \oplus K) = (\operatorname{pf} J)(\operatorname{pf} K)$.

PROOF. $*'' \circ \bigwedge^{p+q}(J \oplus K)$ interchanges the summands $\bigwedge^r E \otimes \bigwedge^s F$ and $\bigwedge^{2p-r} E \otimes \bigwedge^{2q-s} F$ of $\bigwedge^{p+q}(E \oplus F)$, so that only $\bigwedge^p E \otimes \bigwedge^q F$ plays a role in commuting the trace. Hence

$$\begin{aligned} \operatorname{tr}(*'' \circ \textstyle\bigwedge^{p+q}(J \oplus K)) &= \operatorname{tr}(\rho \circ *'' \circ \textstyle\bigwedge^{p+q}(J \oplus K) \circ \iota) \\ &= \operatorname{tr}((\rho \circ *'' \circ \iota) \circ (\rho \circ \textstyle\bigwedge^{p+q}(J \oplus K) \circ \iota)) \\ &= (-1)^{pq} \operatorname{tr}((* \otimes *') \circ (\textstyle\bigwedge^p J \otimes \bigwedge^q K)) \\ &= (-1)^{pq} \operatorname{tr}((* \circ \textstyle\bigwedge^p J) \otimes (*' \circ \bigwedge^q K)) \\ &= (-1)^{pq} (\operatorname{tr} * \circ \textstyle\bigwedge^p J)(\operatorname{tr} *' \circ \bigwedge^q K), \end{aligned}$$

and one inserts the $\pm$ sign of Definition 1.1 to obtain the result.

A free module F of rank $2n$ is oriented by ordering a basis $e_1, \cdots, e_{2n}$ up to even permutation, and the usual inner product is given by $\langle e_i, e_j \rangle = \delta_{ij}$. One easily checks for any permutation π of $(1, \cdots, 2n)$ and $S = e_1 \wedge \cdots \wedge e_{2n}$ that $*(e_{\pi 1} \wedge \cdots \wedge e_{\pi n}) = (-1)^n \varepsilon_\pi e_{\pi(n+1)} \wedge \cdots \wedge e_{\pi(2n)}$, where ε_π is the parity of π; this implies

$$*(e_{\pi 1} \wedge e_{\pi 3} \wedge \cdots \wedge e_{\pi(2n-1)}) = (-1)^{n(n+1)/2} \varepsilon_\pi e_{\pi 2} \wedge e_{\pi 4} \wedge \cdots \wedge e_{\pi(2n)}.$$

PROPOSITION 1.8. *If (K^i_j) represents an endomorphism K of a free R-module F with respect to the basis $e_1, \cdots, e_{2n}$, then*

$$\operatorname{pf} K = (-1)^n \sum_{\Pi} \varepsilon_\pi K^{\pi 1}_{\pi 2} \cdots K^{\pi(2n-1)}_{\pi(2n)},$$

where Π is the set of those permutations π of $(1, \cdots, 2n)$ *with* $\pi 2 < \cdots < \pi(2n)$.

PROOF. Let $P \subset \Pi$ be the set of permutations π satisfying both $\pi 1 < \cdots < \pi(2n-1)$ and $\pi 2 < \cdots < \pi(2n)$, for which

$$\{e_{\pi 2} \wedge \cdots \wedge e_{\pi(2n)} \mid \pi \in P\}$$

is a basis of $\bigwedge^n F$, and let T be the set of all permutations of $(1, \cdots, n)$. Then

$$\begin{aligned} (\textstyle\bigwedge^n K)(e_{\pi 2} \wedge \cdots \wedge e_{\pi(2n)}) &= \sum_{i_1, \cdots, i_n} (K^{i_1}_{\pi 2} \cdots K^{i_n}_{\pi(2n)}) e_{i_1} \wedge \cdots \wedge e_{i_n} \\ &= \left\{ \sum_T \varepsilon_\tau K^{\pi(2\tau(1)-1)}_{\pi 2} \cdots K^{\pi(2\tau(n)-1)}_{\pi(2n)} \right\} e_{\pi 1} \wedge \cdots \wedge e_{\pi(2n-1)} + \cdots, \end{aligned}$$

where "$+\cdots$" represents summands $e_{i_1} \wedge \cdots \wedge e_{i_n}$ with

$$\{i_1, \cdots, i_n\} \neq \{\pi 1, \cdots, \pi(2n-1)\}.$$

Hence

$$(* \circ \textstyle\bigwedge^n K)(e_{\pi 2} \wedge \cdots \wedge e_{\pi(2n)})$$
$$= \left\{(-1)^{n(n+1)/2}\varepsilon_\pi \sum_T \varepsilon_\tau K_{\pi 2}^{\pi(2\tau(1)-1)} \cdots K_{\pi(2n)}^{\pi(2\tau(n)-1)}\right\} e_{\pi 2} \wedge \cdots \wedge e_{\pi(2n)} + \cdots,$$

where "$+\cdots$" now represents summands $e_{i_1} \wedge \cdots \wedge e_{i_n}$ with

$$\{i_1, \cdots, i_n\} \neq \{\pi 2, \cdots, \pi(2n)\}.$$

Hence

$$\begin{aligned} \operatorname{tr} * \circ \textstyle\bigwedge^n K &= (-1)^{n(n+1)/2} \sum_{P,T} \varepsilon_\pi \varepsilon_\tau K_{\pi 2}^{\pi(2\tau(1)-1)} \cdots K_{\pi(2n)}^{\pi(2(\tau(n)-1))} \\ &= (-1)^{n(n+1)/2} \sum_{\Pi} \varepsilon_\pi K_{\pi 2}^{\pi 1} \cdots K_{\pi(2n)}^{\pi(2n-1)}. \end{aligned}$$

We remark that except for a constant factor $\pm 2^n$ Proposition 1.8 states that $\operatorname{pf} K$ is the classical pfaffian of (K_j^i), usually defined only when (K_j^i) is skew-symmetric. If R contains $\frac{1}{2}$ then Proposition 1.8 implies that $\operatorname{pf} K = \operatorname{pf} \frac{1}{2}(K - {}^tK)$, so that $\operatorname{pf} K$ depends only on the skew-symmetric part of K. For certain ground rings R, including the one considered in the next section, a localization procedure can be applied to obtain the same result for any oriented R-module F with inner product.

Assume once more that F is free of rank $2n$, and let $J: F \to F$ have matrix representation consisting of blocks $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ down the main diagonal; then $\det J = 1$, and Proposition 1.7 implies $\operatorname{pf} J = \pm 2^n$. Since any skew-symmetric matrix (K) is of the form $(A \circ J \circ {}^tA)$ for some endomorphism A, Corollary 1.6 becomes $(\operatorname{pf} K)^2 = 2^{2n} \det K$. As in the preceding paragraph this result extends to the skew-symmetric endomorphisms K of any oriented module F with inner product over certain ground rings R.

2. **Euler classes.** For any smooth manifold M let $\mathscr{E}$ be the $C^\infty(M)$-module of differentials on M, i.e., the smooth sections of the cotangent bundle, let $(\bigwedge^* \mathscr{E}, d)$ and $H^*(M)$ be the de Rham complex and de Rham cohomology of M, and let R be the even-dimensional subring $\bigwedge^{2*} \mathscr{E} \subset \bigwedge^* \mathscr{E}$. If $\mathscr{F}$ is the $C^\infty(M)$-module of smooth sections of any oriented $2n$-plane bundle ξ over M, then any connection in ξ may be regarded as a real linear map $D: \bigwedge^* \mathscr{E} \otimes \mathscr{F} \to \bigwedge^* \mathscr{E} \otimes \mathscr{F}$ for which the curvature is a $\bigwedge^* \mathscr{E}$-module endomorphism $D \circ D$ of degree 2; it follows that the curvature induces an R-linear endomorphism $K: F \to F$ for $F = R \otimes \mathscr{F}$. The orientation of ξ induces an orientation of F, and any riemannian metric on ξ induces an inner product on F.

In case (K_j^i) represents the endomorphism induced by K for some trivialization of ξ over an open $U \subset M$, then the Euler class $e(\xi) \in H^{2n}(M)$ is represented by a closed $2n$-form whose restriction to U is $(-4\pi)^{-n} \sum_\Pi \varepsilon_\pi K_{\pi 2}^{\pi 1} \cdots K_{\pi(2n)}^{\pi(2n-1)}$. (See [5, e.g.].) It follows from Proposition

1.8 that $e(\xi)$ is globally represented by $(4\pi)^{-n}$ pf K. In fact one can easily develop geometric Euler classes ab initio in terms of the properties of pf, as established in the preceding section, without reference to the trivializations of ξ; we sketch such a development.

The inner product on F induces a corresponding bilinear form $\langle\ ,\ \rangle$ for the 2-sided $\Lambda^* \mathscr{E}$-module $\Lambda^* \mathscr{E}\otimes\mathscr{F}$, and a connection D is *metric* whenever $\langle De,f\rangle+\langle e, Df\rangle=d\langle e,f\rangle$. To verify that a given D is metric it suffices to suppose that e, f are of the forms $1\otimes s$, $1\otimes t$ for sections s, t of ξ; for convenience we write s, t in place of $1\otimes s$, $1\otimes t$.

LEMMA 2.1. *For any invertible endomorphism A of F let $\langle\ ,\ \rangle'$ be the inner product given by $\langle e,f\rangle'=\langle Ae, Af\rangle$, and extend A to an endomorphism of $\Lambda^* \mathscr{E}\otimes\mathscr{F}$. Then if D is a metric connection with respect to $\langle\ ,\ \rangle$ it follows that $A^{-1}\circ D\circ A$ is a metric connection with respect to $\langle\ ,\ \rangle'$; furthermore the curvature K' of D' is given by $K'=A^{-1}\circ K\circ A$.*

PROOF. For $\theta\in\Lambda^p \mathscr{E}$ and $s, t\in F$ one has

$$\begin{aligned} D'(\theta s) &= A^{-1}D(\theta(As)) = A^{-1}(d\theta \cdot As) + (-1)^p\theta(A^{-1}\circ D\circ A)s \\ &= d\theta\cdot s + (-1)^p\theta\cdot D's, \end{aligned}$$

so that D' is a connection. Next,

$$\begin{aligned} \langle D's, t\rangle' + \langle s, D't\rangle' &= \langle A(A^{-1}DA)s, At\rangle + \langle As, A(A^{-1}DA)t\rangle \\ &= \langle D(As), At\rangle + \langle As, D(At)\rangle \\ &= d\langle As, At\rangle = d\langle s, t\rangle' \end{aligned}$$

so that D' is metric with respect to $\langle\ ,\ \rangle'$. Finally, $K's=D'D's=(A^{-1}DDA)s=(A^{-1}KA)s$.

LEMMA 2.2. *For the metric connections D, D' of Lemma* 2.1 *it follows that* pf $K=$pf$'$ $K'\in\Lambda^{2n} \mathscr{E}$, *where* pf$'$ *is defined with respect to* $\langle\ ,\ \rangle'$.

PROOF. Lemmas 1.4 and 2.1 give $*'\circ\Lambda^n K'=(\Lambda^n A^{-1}\circ *\circ\Lambda^n A)\circ(\Lambda^n A^{-1}\circ\Lambda^n K\circ\Lambda^n A)=(\Lambda^n A)^{-1}\circ(*\circ\Lambda^n K)\circ(\Lambda^n A)$, hence

$$\mathrm{tr}\, *'\circ\Lambda^n K' = \mathrm{tr}\, *\circ\Lambda^n K.$$

Now for any oriented $2n$-plane bundle ξ one imposes a riemannian metric on ξ and computes pf $K\in\Lambda^{2n} \mathscr{E}$ for any connection D. The Chern-Weil theorem guarantees that pf K is closed and that the cohomology class [pf K]$\in H^{2n}(M)$ is independent of D. (The Chern-Weil theorem can be presented entirely in terms of algebraic operations on $\Lambda^* \mathscr{E}\otimes\mathscr{F}$; see **[6]**, e.g.) To show that [pf K] is also independent of the metric one observes that any two inner products on F are related by $\langle e,f\rangle' = \langle Ae, Af\rangle$ for some invertible endomorphism $A: F \to F$; hence if D is a metric connection with respect to $\langle\ ,\ \rangle$ Lemma 2.1 provides a metric connection D' with respect to $\langle\ ,\ \rangle'$ for which Lemma 2.2 gives $[\mathrm{pf}' K'] = [\mathrm{pf}\, K]$ as desired. Thus the cohomology class $(4\pi)^{-n}[\mathrm{pf}\, K] \in H^{2n}(M)$ depends only on ξ itself; it will be denoted $e(\xi)$. The construction of $\mathrm{pf}\, K$ guarantees naturality of $e(\xi)$, Proposition 1.7 provides the product formula $e(\xi \oplus \eta) = e(\xi)e(\eta)$, and one verifies normalization as in **[5]**, thus completing the axiomatic characterization of Euler classes.

The distinguishing feature of the preceding sketch is that one proves explicitly that $[\mathrm{pf}\, K]$ is independent of the riemannian metric on ξ. The same technique gives an instant proof of the Avez-Chern theorem (see **[1]**, **[2]**, and **[3]**), in which one imposes a pseudo-riemannian metric on ξ. We remark that a metric of type (p,q) on ξ induces an inner product $\langle\ ,\ \rangle'$ on F with $\langle S, S\rangle' = (-1)^q$ for any orientation $S \in \bigwedge^{2n} F$; this alters none of the methods of this note. Here is the Avez-Chern theorem:

PROPOSITION 2.3. *Given a pseudo-riemannian metric on an oriented* $2n$*-plane bundle* ξ*, let* $\langle\ ,\ \rangle'$ *be the induced inner product on* F *and compute* pf' *with respect to* $\langle\ ,\ \rangle'$*; then the Euler class* $e(\xi)$ *is represented by* $(4\pi)^{-n}\,\mathrm{pf}' K'$ *for the curvature* K' *of any connection* D' *which is metric with respect to* $\langle\ ,\ \rangle'$.

PROOF. $\langle\ ,\ \rangle'$ induces an inner product $\langle\ ,\ \rangle' : F_C \times F_C \to R_C$ in the complexification F_C of F. (Symmetric, not hermitian symmetric.) Let $\langle\ ,\ \rangle : F_C \times F_C \to R_C$ be the inner product induced by any riemannian metric in ξ, so that $\langle e,f\rangle' = \langle Ae, Af\rangle$ for some invertible $A \in \mathrm{Hom}_{R_C}(F_C, F_C)$. If $*$ and $*'$ are Hodge operators for $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ one has $*' = \bigwedge^n A^{-1} \circ * \circ \bigwedge^n A$ as in Lemma 1.4, and $A \circ D' \circ A^{-1}$ is a metric connection D with respect to $\langle\ ,\ \rangle$ as in Lemma 2.1. Then $\mathrm{pf}' K' = \mathrm{pf}\, K$ as in Lemma 2.2, which completes the proof.

We remark that Corollary 1.6 has not been used in this section. However, its sharpened form $(\mathrm{pf}\, K)^2 = 2^{2n} \det K$ has a well-known application: $e(\xi)^2 = p_n(\xi)$.

REFERENCES

1. A. Avez, *Formule de Gauss-Bonnet-Chern en métrique de signature quelconque*, C. R. Acad. Sci. Paris **255** (1962), 2049–2051; Rev. Un. Mat. Argentina **21** (1963), 191–197. MR **26** #2993; **29** #6452.

2. A. Borel, *Sur une généralisation de la formule de Gauss-Bonnet*, An. Acad. Brasil. Ci. **39** (1967), 31–37. MR **36** #5866.

3. S. S. Chern, *Pseudo-Riemannian geometry and the Gauss-Bonnet formula*, An. Acad. Brasil. Ci. **35** (1963), 17–26. MR **27** #5196.

4. H. Flanders, *On free exterior powers*, Trans. Amer. Math. Soc. **145** (1969), 357–367. MR **40** #2662.

5. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. II, Interscience Tracts in Pure and Appl. Math., no. 15, Interscience, New York, 1969. MR **38** #6501.

6. H. Osborn, *Differential geometry in PL* (to appear).

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