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SOLUTIONS OF $(ry^{(n)})^{(n)}+qy=0$ OF CLASS $\mathscr{L}_p[0, \infty)$

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ABSTRACT. For a certain class of ordinary differential operators L, this paper determines the maximum number m of linearly independent solutions of class $\mathscr{L}_p[0, \infty)$ of $L(y)=0$. For $L(y)=(ry^{(n)})^{(n)}+qy$ and $p=2$, the principal result is that if $\int_0^t |q|^2\, d\tau=O(t)$ as $t\to\infty$, then $m\leqq n$.

We consider first the differential operators

$$(1)\quad L_1(y) = (-1)^n(ry^{(n)})^{(n)} + qy \quad\text{and}\quad L_2(z) = (-1)^n(rz^{(n)})^{(n)} + \bar{q}z,$$

where r and q are continuous, complex-valued functions on $[0, \infty)$ with r positive. A result of A. Zettl [5] states that if q is real and $\int_0^\infty q^2\, dt<\infty$, then the equation $L_1(y)=0$ has a solution which is not of class $\mathscr{L}_2[0, \infty)$. In this note we investigate the number of $\mathscr{L}_p[0, \infty)$ $(p>1)$ solutions of (1) under a less restrictive hypothesis than that used by A. Zettl. However, we do not obtain results for the more general differential equations considered in [5].

Define s by $(1/p)+(1/s)=1$,

$$V_1 = \{y \mid L_1(y) = 0 \text{ and } y \in \mathscr{L}_p[0, \infty)\},$$

and

$$V_2 = \{z \mid L_2(z) = 0 \text{ and } z \in \mathscr{L}_p[0, \infty)\}.$$

By taking conjugates in (1), it follows that the correspondence $y\to\bar{y}$ is one-one from V_1 onto V_2; hence $\dim V_1=\dim V_2$.

THEOREM 1. *If for some* $K>0$,

$$(2)\quad \left(\int_0^t |q|^s\, d\tau\right)^{1/s} \leqq Kt^{1-1/s} \quad \textit{for } t \geqq 1,$$

then $\dim V_1=\dim V_2\leqq n$.

PROOF. Let $y\in V_1$; then as in [5], by Hölder's inequality,

$$|(ry^{(n)})^{(n-1)}(t) - (ry^{(n)})^{(n-1)}(1)| = \left|\int_1^t qy\, d\tau\right| \leqq \left(\int_1^t |q|^s\, d\tau\right)^{1/s}\left(\int_1^t |y|^p\, d\tau\right)^{1/p}.$$

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Thus (2) implies there is a number K_1, depending on y, such that

$$|(ry^{(n)})^{(n-1)}(t)| \leqq K_1 t^{1-1/s} \quad \text{for } t \geqq 1. \tag{3}$$

Similarly, for $z \in V_2$ there is a number K_2, depending on z, such that

$$|(rz^{(n)})^{(n-1)}(t)| \leqq K_2 t^{1-1/s} \quad \text{for } t \geqq 1. \tag{4}$$

For $L_1(y)=0$ and $L_2(z)=0$, the Lagrange identity $[y, z]'=0$ holds where

$$[y, z] = \sum_{i=0}^{n-1} (-1)^i [y^{(i)}(r\bar{z}^{(n)})^{(n-1-i)} - (ry^{(n)})^{(n-1-i)}\bar{z}^{(i)}].$$

Suppose now dim $V_1 > n$. We will show that for some $y \in V_1$ and $z \in V_2$, $[y, z]=1$.

Let $V=\{y|L_1(y)=0\}$. Define the linear transformation S from V_2 into V by $S(z)=w$ means

$$z^{(i)}(0) = w^{(i)}(0) \quad \text{and} \quad (rz^{(n)})^{(i)}(0) = (rw^{(n)})^{(i)}(0), \qquad i = 0, \cdots, n-1.$$

Since solutions of linear equations are uniquely determined by their initial values, S is one-one and dim $S(V_2)=$dim V_2. Define the linear transformation T from V_1 into V by $T(y)=w$ means for $i=0, \cdots, n-1$,

$$y^{(i)}(0) = (-1)^i(rw^{(n)})^{(n-1-i)}(0) \quad \text{and} \quad (ry^{(n)})^{(n-1-i)}(0) = (-1)^{i+1}w^{(i)}(0).$$

Then T is nonsingular; hence dim $T(V_1)=$dim V_1. Since dim $V_1=$dim $V_2>n$ and dim $V=2n$, there is a $w\neq 0$, $w\in T(V_1)\cap S(V_2)$. Let $y=T^{-1}(w)$ and $z=S^{-1}(w)$. Then

$$[y, z] = \sum_{i=0}^{n-1} [(rw^{(n)})^{(n-1-i)}(r\bar{w}^{(n)})^{(n-1-i)} + w^{(i)}\bar{w}^{(i)}]\big|_{t=0} > 0. \tag{5}$$

Multiplication of (5) by an appropriate constant yields $[y, z]=1$ for some $y\in V_1$ and $z \in V_2$.

For $a>1$, it follows from $[y, z]=1$ that

$$\begin{aligned} \frac{t}{n+1}\left[1-\frac{a}{t}\right]^{n+1} &= \int_a^t \left(1-\frac{\tau}{t}\right)^n d\tau \\ &= \sum_{i=0}^{n-1}\int_a^t (-1)^i[y^{(i)}(r\bar{z}^{(n)})^{(n-1-i)} - (ry^{(n)})^{(n-1-i)}\bar{z}^{(i)}]\left(1-\frac{\tau}{t}\right)^n d\tau. \end{aligned} \tag{6}$$

An i-fold integration by parts gives

$$\begin{aligned} &\int_a^t y^{(i)}(r\bar{z}^{(n)})^{(n-1-i)}\left(1-\frac{\tau}{t}\right)^n d\tau \\ &\quad = M_{i,t} + (-1)^i\int_a^t y\,\frac{d^{(i)}}{d\tau^{(i)}}\left[(r\bar{z}^{(n)})^{(n-1-i)}\left(1-\frac{\tau}{t}\right)^n\right] d\tau \end{aligned} \tag{7}$$

where $M_{i,t}$ satisfies $|M_{i,t}|=O(1)$ as $t\to\infty$. From (4) we conclude that for some $N_1>0$ and $k=1,\cdots,n$,

$$|(r\bar{z}^{(n)})^{(n-k)}(t)| \leqq N_1 t^{k-1/s} \quad \text{for } t \geqq 1. \tag{8}$$

An application of Leibniz's rule and (8) yields an $N_2>0$, depending only on z, such that

$$\left|\frac{d^{(i)}}{d\tau^{(i)}}\left[(r\bar{z}^{(n)})^{(n-1-i)}\left(1-\frac{\tau}{t}\right)^n\right]\right| \leqq N_2\tau^{1-1/s} \quad \text{for } 1 \leqq \tau \leqq t.$$

Applying this inequality to (7), we obtain

$$\begin{aligned}&\left|\int_a^t y^{(i)}(r\bar{z}^{(n)})^{(n-1-i)}\left(1-\frac{\tau}{t}\right)^n d\tau\right| \\ &\quad\leqq |M_{i,t}| + N_2\left(\int_a^t |y|^p\, d\tau\right)^{1/p}\left(\int_a^t \tau^{s-1}\, d\tau\right)^{1/s}.\end{aligned} \tag{9}$$

Since $(\int_a^t \tau^{s-1}\, d\tau)^{1/s} \leqq t/(s)^{1/s}$, we have from inequality (9) that

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t}\sum_{i=0}^{n-1}\left|\int_a^t y^{(i)}(r\bar{z}^{(n)})^{(n-1-i)}\left(1-\frac{\tau}{t}\right)^n d\tau\right| \\ &\quad\leqq nN_2\left(\int_a^\infty |y|^p\, d\tau\right)^{1/p}\Big/(s)^{1/s}.\end{aligned} \tag{10}$$

Since N_2 does not depend on a, the right-hand side of (10) is $<1/4(n+1)$ for sufficiently large a. In a similar manner we may prove that with y and z interchanged, the limit superior as $t\to\infty$ of (10) is $<1/4(n+1)$. However, this gives a contradiction to (6) since

$$\operatorname*{limit}_{t\to\infty} \frac{1}{t}\left\{\frac{t}{n+1}\left[1-\frac{a}{t}\right]^{n+1}\right\} = \frac{1}{n+1}.$$

As a corollary to Theorem 1 we obtain the deficiency indices of the formally selfadjoint operator L where

$$L(y) = (ry^{(n)})^{(n)} + Qy. \tag{11}$$

COROLLARY. *Let Q be a real continuous function on* $[0, \infty)$ *such that* $\int_0^t Q^2\, d\tau = O(t)$ *as* $t\to\infty$. *If L is given by* (11), *then the equation* $L(y)=\lambda y$, $\operatorname{Im} \lambda\neq 0$, *has exactly n linearly independent solutions of class* $\mathscr{L}_2[0, \infty)$.

PROOF. If we choose $q=Q-\lambda$, then an application of Theorem 1 gives that $L(y)=\lambda y$ has at most n linearly independent solutions of class $\mathscr{L}_2[0, \infty)$. However, the classical theory of deficiency indices (cf. [3, Chapter V]) gives that $L(y)=\lambda y$, $\operatorname{Im} \lambda\neq 0$, has at least n linearly independent solutions of class $\mathscr{L}_2[0, \infty)$ with no restrictions on the growth of Q.

For $p=2$, additional results are given in [2] under which dim $V_1 \leqq n$. The conditions on q in [2] are in the form of boundedness conditions on the growth of q and are independent of the conditions considered here.

The arguments above may be adapted to certain odd-order equations. For example, with r and q as before, define

$$L_3(y) = (ry')'' + qy \quad \text{and} \quad L_4(z) = (rz'')' - \bar{q}z.$$

Then $L_3(y)=0$ and $L_4(z)=0$ gives $[y, z]'=0$ where

$$[y, z] = y(r\bar{z}'') - (ry')\bar{z}' + (ry')'\bar{z}.$$

If $V_3=\{y|L_3(y)=0 \text{ and } y\in\mathscr{L}_p[0, \infty)\}$, $V_4=\{z|L_4(z)=0 \text{ and } z\in\mathscr{L}_p[0, \infty)\}$, and dim V_3+dim $V_4>3$, we may as before show that for some $y\in V_3$ and $z\in V_4$, $[y, z]=1$. If q satisfies $\int_0^t |q|^s\, d\tau=O(t^{s-1})$ as $t\to\infty$, then for $y\in V_3$ and $z\in V_4$ we have $|(ry')'(t)|$ and $|(rz'')(t)|$ are $O(t^{1-1/s})$ as $t\to\infty$. From

$$\begin{aligned}\frac{t}{2}\left(1-\frac{a}{t}\right)^2 &= \int_a^t \left(1-\frac{\tau}{t}\right) d\tau \\ &= \int_a^t \{y(r\bar{z}'') - (ry')\bar{z}' + (ry')'\bar{z}\}\left(1-\frac{\tau}{t}\right) d\tau,\end{aligned}$$

we obtain, after integrating $\int_a^t (ry')\bar{z}'(1-\tau/t)\,d\tau$ by parts, a contradiction as in the proof of Theorem 1. This gives dim V_3+dim $V_4\leqq 3$. However, in general, dim $V_3\neq$dim V_4, e.g., consider $r=1$, $q=-1-i$.

We note also that the proof given by A. Zettl may be adapted to more general equations. For example, let

$$L_5(y)=[(ry'')' + py']' + qy$$

where r and q are as before and p is a continuous, complex-valued function on $[0, \infty)$. We now show that if $V_5=\{y|L_5(y)=0 \text{ and } y\in\mathscr{L}_p[0, \infty)\}$, then the condition $\int_0^t |q|^s\, d\tau=O(t^{s-1})$ as $t\to\infty$ implies that dim $V_5\leqq 3$.

Suppose to the contrary that dim $V_5=4$. Choose solutions y_1, y_2, y_3, y_4 of $L_5(y)=0$ so that $Y(0)=I$ where

$$Y = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ ry_1'' & ry_2'' & ry_3'' & ry_4'' \\ (ry_1'')' + py_1' & (ry_2'')'+ py_2' & (ry_3'')' + py_3' & (ry_4'')' + py_4' \end{bmatrix}.$$

The Wronskian W=det Y is constant and expanding W along the last row of Y we obtain

$$1 = W = \sum_{i=1}^{4} Z_i[(ry_i'')' + py_i'] \tag{12}$$

where Z_i is the cofactor of $(ry_i'')'+py_i'$. By differentiation it may be shown that $Z_i=(-1)^i y_{5-i}$. As above we may show that for each i,

$$|[(ry_i'')' + py_i'](t)| = O(t^{1-1/s}) \quad \text{as } t \to \infty. \tag{13}$$

From equation (12) we obtain for $a>0$,

$$t - a = \int_a^t W\, d\tau \leqq \sum_{i=1}^{4} \left(\int_a^t |y_{5-i}|^p\, d\tau\right)^{1/p} \left(\int_a^t |(ry_i'')' + py_i'|^s\, d\tau\right)^{1/s}. \tag{14}$$

Application of (13) in (14) now yields a contradiction as in the proof of Theorem 1.

Without additional assumptions on r and p, it is possible to have dim $V_5=3$. W. N. Everitt **[1]** mentions the example of A. D. Wood,

$$y^{(\mathrm{iv})} + (t^2y')' + y = \lambda y, \qquad \operatorname{Im} \lambda \neq 0,$$

which has three linearly independent solutions of class $\mathscr{L}_2[0, \infty)$. Results of P. Walker **[4]** show that the equation,

$$(t^\alpha y'')'' + (t^\beta y')' \pm t^\gamma y = \lambda y, \qquad \operatorname{Im} \lambda \neq 0,$$

has three linearly independent solutions of class $\mathscr{L}_2[1, \infty)$ if $\alpha<4$, $\beta>2$, and $\gamma\leqq 0$.

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