PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, March 1972

AN INVARIANT SUBSPACE THEOREM

MARY R. EMBRY

ABSTRACT. A continuous linear operator on a complex Banach space of dimension greater than 1 which is strictly cyclic with respect to its commutant is shown to have a nontrivial closed invariant subspace.

Consider a complex Banach space X of dimension greater than 1. A continuous linear operator A on X is *strictly cyclic* with respect to its commutant $(A)'$ if there exists an element x of X such that $\{Sx:S\in(A)'\}=X$. An *invariant subspace* of A is a closed linear subspace M of X such that $M\neq\{0\}$, $M\neq X$ and $A(M)\subset M$.

THEOREM. *If A is an operator on X which is strictly cyclic with respect to its commutant, then either*

(i) *the null space of some nonzero element of $(A)'$ is nonzero, or*

(ii) *the range of each noninvertible element of $(A)'$ is nondense.*

Thus A has an invariant subspace.

PROOF. The final assertion follows immediately from (i) and (ii). Assume that $\{Sx:S\in(A)'\}=X$ and that (i) does not occur. Then each nonzero element of $(A)'$ is one-to-one and hence the mapping $S\rightarrow Sx$ of $(A)'$ into X is a continuous one-to-one linear mapping of $(A)'$ onto X. The Open Mapping Theorem implies the existence of a positive number K such that

$$K\,\|S\| \leqq \|Sx\| \quad \text{for all } S \text{ in } (A)'. \tag{1}$$

Now let T be a noninvertible element of $(A)'$ and let N be the closure of the range, $T(X)$, of T. Since $\{Sx:S\in(A)'\}=X$, $\{TSx:S\in(A)'\}=T(X)$. Thus if $N=X$, there exists a sequence $\{S_n\}$ of elements of $(A)'$ such that $\lim_{n\to\infty} TS_nx=x$. Inequality (1) now implies that $\lim_{n\to\infty}\|TS_n-I\|=0$, where I is the identity operator on X. Therefore, if n is sufficiently large, TS_n is invertible and hence T is onto. We now have T one-to-one and onto. A second application of the Open Mapping Theorem tells us that T is invertible, contradicting our choice of T. Thus $N\neq X$ and the proof is complete.

Received by the editor June 18, 1971.
AMS 1970 *subject classifications*. Primary 47A15.
Key words and phrases. Banach space, commutant, strictly cyclic, invariant subspace.

The existence of an invariant subspace of A under the hypotheses of the theorem for the case in which X is a Hilbert space is shown in [1]. However, the theorem appears to be a new result in the general Banach space setting.

REFERENCES

1. Alan Lambert, *Strictly cyclic operator algebras*, Ph.D. Thesis, University of Michigan, Ann Arbor, Mich., 1970.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE, CHARLOTTE, NORTH CAROLINA 28213