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# A NONPARACOMPACT SPACE WHICH ADMITS A CLOSURE-PRESERVING COVER OF COMPACT SETS

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ABSTRACT. In **[1]**, Tamano asked whether or not a space which is the union of a closure-preserving family of compact subsets must be paracompact. It is the purpose of this paper to present a space which admits such a cover, is completely regular and $T_2$, has a basis consisting of open and closed sets, yet fails to be even normal.

In this paper, we present an example of a completely regular, $T_2$ space which has a closure-preserving cover consisting of compact sets, yet fails to be normal. Let $W=\{\sigma|\sigma<\omega_1\}$ be the collection of countable ordinals. Let $T=\{(\sigma, \lambda)|\sigma, \lambda\in W, \sigma<\lambda\}$. Let $X=W\cup T$.

We define a topology on $X$ as follows: For each $\sigma\in W$, a set $V$ containing $\sigma$ shall be a basic open set about $\sigma$ provided:

(1) $V$ contains no ordinal other than $\sigma$ itself.

(2) $V$ contains all but finitely many members of the set $\{(\sigma, \lambda)|\lambda>\sigma\}$.

(3) $V$ contains all but finitely many members of the set $\{(\lambda, \sigma)|\lambda<\sigma\}$.

For each $(\sigma, \lambda)\in T$, the singleton set $\{(\sigma, \lambda)\}$ is open.

It is clear that the sets described above constitute a basis for a topology on $X$. This topology is completely regular and $T_2$ and, in fact, has a basis consisting of open and closed sets.

To see this, let $V$ be a basic open set about an ordinal $\sigma_1$. The only candidates for limit points of $V$, not already in $V$, are other ordinal numbers. Let $\sigma_2\in W$, with $\sigma_2\neq\sigma_1$, and suppose $\sigma_2>\sigma_1$. Let $U=\{\sigma_2\}\cup\{(\lambda, \sigma_2)|\lambda<\sigma_2, \lambda\neq\sigma_1\}\cup\{(\sigma_2, \lambda)|\lambda>\sigma_2\}$. Then $U$ is a basic open set about $\sigma_2$, and $U\cap V=\varnothing$. For if $U\cap V\neq\varnothing$, this would require either an ordinal $\lambda>\sigma_2$ such that $(\sigma_2, \lambda)\in V\cap U$, which is not possible, since every member of $V$ has first co-ordinate $\sigma_1$, or $U\cap V\neq\varnothing$ would require an ordinal $\lambda<\sigma_2$ such that $(\lambda, \sigma_2)\in V\cap U$. Since every member of $V$ has first co-ordinate $\sigma_1$, this would mean that $(\lambda, \sigma_2)=(\sigma_1, \sigma_2)$, whence, $\lambda=\sigma_1$. But $(\sigma_1, \sigma_2)$ is not a member of $U$, so that $U\cap V=\varnothing$. Thus no ordinal $\sigma_2>\sigma_1$, can be a limit point of $V$. For $\sigma_2<\sigma_1$, let $U=\{\sigma_2\}\cup\{(\lambda, \sigma_2)|\lambda<\sigma_2\}\cup\{(\sigma_2, \lambda)|\lambda>\sigma_2, \lambda\neq\sigma_1\}$. The same type of argument as

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above shows that $U\cap V=\varnothing$, and $V$ is now seen to be closed. Thus $X$ has a basis consisting of open and closed sets.

But $X$ is not normal, hence not paracompact. In fact, $X$ is not even weakly normal, that is, given two disjoint closed sets, one of which is countable, it still need not be true that the closed sets have disjoint neighborhoods. To see this, let $A=\{\sigma_i|i=1,2,\cdots\}$ be any countable subset of $W$, and let $B=\{\lambda|\lambda>\sup A\}$. Both $A$, $B$ are readily seen to be closed.

Now let $V$ be an open set about $A$. For each $i\in Z^+$, let $V(i)$ be a basic open set such that $\sigma_i\in V(i)\subset V$. For each $i\in Z^+$, $V(i)$ contains all but finitely many points of the set $\{(\sigma_i,\lambda)|\lambda>\sigma_i\}$. For each $i\in Z^+$, let $\{\lambda_{i,1},\lambda_{i,2},\cdots,\lambda_{i,k_i}\}$ be a listing of those ordinals for which $(\sigma_i,\lambda)$ is not a member of $V(i)$. Let $\alpha$ be any member of $B$, with $\alpha$ larger than

$$\sup\{\lambda_{i,j}\mid i=1,2,\cdots,i\leqq j\leqq k_i\}.$$

Then $\alpha$ is a limit point of $V$, for if $U$ is a basic open set about $\alpha$, then $U$ contains all points of the set $\{(\lambda,\alpha)|\lambda<\alpha\}$, with at most a finite number of exceptions. In particular, then, there is an integer $t$ such that $(\sigma_t,\alpha)\in U$. But since $\alpha>\sup\{\lambda_{i,j}|i=1,2,\cdots,1\leqq j\leqq k_i\}$, then $(\sigma_t,\alpha)$ is not one of the points excluded from $V(t)$, whence $(\sigma_t,\alpha)\in V(t)\subset V$, and we have $V\cap U\neq\varnothing$.

Thus the closure of every open set about $A$ picks up points of $B$, whence $A$, $B$ cannot have disjoint neighborhoods, and $X$ is not normal, nor even weakly normal.

It may be worth noting that $X$ is metacompact, for if $V(\sigma)=\{\sigma\}\cup\{(\sigma,\lambda)|\lambda>\sigma\}\cup\{(\lambda,\sigma)|\lambda<\sigma\}$, then the family $\{V(\sigma)|\sigma\in W\}$ is readily seen to be a point-finite open covering of $X$, with each set $V(\sigma)$ metacompact, in fact compact.

But $X$ has a closure-preserving cover, each member of which is compact. To see this, we proceed as follows: For each $(\sigma,\lambda)\in T$, let $F(\sigma,\lambda)=\{\sigma,\lambda,(\sigma,\lambda)\}$. The family $\{F(\sigma,\lambda)|(\sigma,\lambda)\in T\}$ is certainly a cover of $X$, and each member of the cover has three elements, hence is surely compact. To show that this cover is closure-preserving, let $\hat{T}\subset T$. Any isolated point of $X$ which appears in

$$\overline{\bigcup\{F(\sigma,\lambda)\mid(\sigma,\lambda)\in\hat{T}\}}$$

is clearly already in the set $\bigcup\{F(\sigma,\lambda)|(\sigma,\lambda)\in\hat{T}\}$. The only question is the case of an ordinal $\alpha$ which is a member of

$$\overline{\bigcup\{F(\sigma,\lambda)\mid(\sigma,\lambda)\in\hat{T}\}}.$$

In this case, the set $V(\alpha)=\{\alpha\}\cup\{(\lambda,\alpha)|\lambda<\alpha\}\cup\{(\alpha,\lambda)|\alpha<\lambda\}$ is an open set about $\alpha$, hence must meet $\bigcup\{F(\sigma,\lambda)|(\sigma,\lambda)\in\hat{T}\}$. Let $(\sigma_1,\lambda_1)\in\hat{T}$ such that $V(\alpha)\cap F(\sigma_1, \lambda_1)\neq\varnothing$. Since $F(\sigma_1, \lambda_1)=\{\sigma_1, \lambda_1, (\sigma_1, \lambda_1)\}$, one of these three points must be in $V(\alpha)$. If either $\sigma_1$ or $\lambda_1$ is in $V(\alpha)$, we have $\sigma_1$ or $\lambda_1$ equal to $\alpha$, since $\sigma_1$, $\lambda_1$, are ordinals, and $\alpha$ is the only ordinal which appears in $V(\alpha)$. In either case, $\alpha\in F(\sigma_1, \lambda_1)\subset\bigcup\{F(\sigma, \lambda)|(\sigma, \lambda)\in\hat{T}\}$, as required. If $(\sigma_1, \lambda_1)\in V(\alpha)$, then we have either $(\sigma_1, \lambda_1)=(\lambda, \alpha)$ for some $\lambda$, or else $(\sigma_1, \lambda_1)=(\alpha, \lambda)$ for some $\lambda$. We have in the first case that $\alpha=\lambda_1$, and in the second case that $\alpha=\sigma_1$, hence in either case, $\alpha\in\{\sigma_1, \lambda_1, (\sigma_1, \lambda_1)\}=F(\sigma_1, \lambda_1)\subset\bigcup\{F(\sigma, \lambda)\in\hat{T}\}$, as required. This proves that the covering is closure preserving.

## Reference

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