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ON SEMILOCAL OP-RINGS

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ABSTRACT. The notion of OP-rings was introduced by D. Lissner. A commutative ring R is called an OP-ring if, for any $n \geqq 2$, any vector of R^n is an outer product of $n-1$ vectors of R^n. Recently J. Towber proved that any local ring is an OP-ring if and only if the maximal ideal is generated by two elements. The main result in the present paper is a generalization to semilocal rings of the above theorem proved by Towber for local rings. The author's argument does not rely on Towber's theorem however, and so provides a new and very elementary proof of that result.

In this note, all rings are commutative rings with 1. R^n denotes a free module of rank n over R. For an $(n-1) \times n$ matrix M with coefficients in R, we denote by $|M| = (b_1, \cdots, b_n)$ a vector in R^n where $b_i = (-1)^{i+1}$ (the determinant of the matrix obtained by deleting the ith column from M). We say that a vector v in R^n is an outer product if there exists an $(n-1) \times n$ matrix M such that $|M| = v$. We say that a ring R is an OP-ring if, for any $n \geqq 2$, any vector in R^n is an outer product. If $v = (a_1, \cdots, a_n)$ is a vector in R^n, we write $Iv = I(a_1, \cdots, a_n)$ for the ideal generated by $a_1, \cdots, a_n$. If $Iv = 1$, we call v a unimodular vector. When $v_1, \cdots, v_s \in R^n$, we denote by $\|v_1, \cdots, v_s\|$ the matrix the ith row of which is v_i. For a matrix M, ${}^tM$ denotes the transpose of M. $\mu(N)$ denotes the minimal number of generators of a module N.

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1. The objective in this section is to prove

THEOREM. *Let R be a (noetherian) semilocal ring with the maximal ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$. Then R is an OP-ring if and only if $\mu(\mathfrak{m}_i R_{\mathfrak{m}_i}) \leqq 2$ for each i.*

The "only if" part is Theorem 2.4 (and the remark following that theorem) of [9]. To prove the "if" part we need some lemmas.

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LEMMA 1. *Let R be a semilocal ring with the maximal ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$. Suppose that $\mathfrak{m}_i R_{\mathfrak{m}_i}$ is generated by r elements for each i. Then there exist $r+s-1$ elements $x_1, \cdots, x_{r-1}, y_1, \cdots, y_s$ of R such that $\mathfrak{m}_i = I(x_1, \cdots, x_{r-1}, y_i)$, $x_j \in J$, for each i, j where J is the Jacobson radical of R, i.e., $J = \bigcap \mathfrak{m}_i$.*

PROOF. Assume that $m_{i1}, \cdots, m_{ir} \in \mathfrak{m}_i$, $\mathfrak{m}_i R_{\mathfrak{m}_i} = (m_{i1}, \cdots, m_{ir}) R_{\mathfrak{m}_i}$ for each i. Since $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$ are distinct maximal ideals, we have $\mathfrak{m}_i^2 \not\subseteq \mathfrak{m}_j$ for $i \neq j$. Hence, by virtue of Proposition 2 of Chapter II, §1 of [**1**], there exist s elements $e_1, \cdots, e_s$ of R such that $e_i \in \mathfrak{m}_i^2$, $e_i \notin \mathfrak{m}_j$ for each $j \neq i$. Put $d_i = \prod_{j \neq i} e_j$, and we have $d_i \notin \mathfrak{m}_i$, $d_i \in \mathfrak{m}_j^2$ for each $j \neq i$. Put $x_j = \sum_i m_{ij} d_i$ for $j < r$, $y_j = m_{jr} d_j + e_j$, and we have that $x_j \in J$, $y_j \notin \mathfrak{m}_k$ for each $k \neq j$. Now we see that $\mathfrak{m}_j = I(x_1, \cdots, x_{r-1}, y_j)$. For:

$$\begin{aligned} \mathfrak{m}_j R_{\mathfrak{m}_j} / \mathfrak{m}_j^2 R_{\mathfrak{m}_j} &= I(\bar{m}_{j1}, \cdots, \bar{m}_{jr}) = I(\bar{m}_{j1} \bar{d}_j, \cdots, \bar{m}_{j,r-1} \bar{d}_j, \bar{y}_j) \\ &= I(\bar{x}_1, \cdots, \bar{x}_{r-1}, \bar{y}_j). \end{aligned}$$

Put $\mathfrak{a}_j = I(x_1, \cdots, x_{r-1}, y_j)$, and we have $\mathfrak{m}_j R_{\mathfrak{m}_j} = (\mathfrak{a}_j + \mathfrak{m}_j^2) R_{\mathfrak{m}_j}$. From this we deduce $\mathfrak{m}_j = \mathfrak{a}_j + \mathfrak{m}_j^2$ i.e., $\mathfrak{m}_j \equiv \mathfrak{m}_j^2 \pmod{\mathfrak{a}_j}$, as follows. Given $x \in \mathfrak{m}_j$ we can write $x/1 = (a+y)/s$ in $R_{\mathfrak{m}_j}$, with $a \in \mathfrak{a}_j$, $y \in \mathfrak{m}_j^2$, and $s \notin \mathfrak{m}_j$. Then there exist $t \notin \mathfrak{m}_j$ such that $tsx = t(a+y)$, and an $r \notin \mathfrak{m}_j$ such that $rts = 1+m$, with $m \in \mathfrak{m}_j$. Then $x - xm^2 = x(rts)(1-m) = rt(a+y)(1-m) \in \mathfrak{a}_j + \mathfrak{m}_j^2$, and $xm^2 \in \mathfrak{m}_j^2$ so $x \in \mathfrak{a}_j + \mathfrak{m}_j^2$; hence $\mathfrak{m}_j = \mathfrak{a}_j + \mathfrak{m}_j^2$. Since $y_j \notin \mathfrak{m}_k$ for $k \neq j$, $R/\mathfrak{a}_j$ is local. Therefore $\mathfrak{m}_j \equiv \mathfrak{m}_j^2 \pmod{\mathfrak{a}_j}$ implies that $\mathfrak{m}_j = \mathfrak{a}_j$.

REMARK. Under the assumption of Lemma 1, it is known that any maximal ideal is generated by r elements (cf. [**2**] and [**8**]).

The following two lemmas are well known. We shall write the proofs for the completeness.

LEMMA 2. *Let R be a semilocal ring with the maximal ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$. Assume that each $\mathfrak{m}_i$ is principal. Then R is a principal ideal ring.*

PROOF. Let $\mathfrak{a}$ be any ideal of R. Since $R_{\mathfrak{m}_i}$ is a principal ideal ring for each i, we put $\mathfrak{a} R_{\mathfrak{m}_i} = a_i R_{\mathfrak{m}_i}$, $a_i \in \mathfrak{a}$. Put $a = \sum a_i d_i$ where d_i, $i = 1, \cdots, s$, are the same as in the proof of Lemma 1. Then we have $a \equiv a_i d_i \pmod{\mathfrak{a}\mathfrak{m}_i}$. Therefore we have $\mathfrak{a} R_{\mathfrak{m}_i} = (aR + \mathfrak{a}\mathfrak{m}_i) R_{\mathfrak{m}_i}$, hence $\mathfrak{a} R_{\mathfrak{m}_i} = a R_{\mathfrak{m}_i}$ for each i by Nakayama's lemma, whence we have $\mathfrak{a} = aR$.

LEMMA 3. *Let R be a principal ideal ring, v a vector of R^n and $Iv = aR$ for an $a \in R$. Then there exists an invertible $n \times n$ matrix M such that $vM = (a, 0, \cdots, 0)$.*

PROOF. Put $v = (a_1, \cdots, a_n)$, $a = \sum a_i b_i$ and $R = \sum \oplus D_j$, $1 = \sum e_j$, $e_j \in D_j$ for each j where D_j is a principal ideal domain or a special principal ideal ring (cf. [**11**]). If $ae_j \neq 0$, we see that $(b_1, \cdots, b_n) e_j$ is unimodular in D_j.

Since all projective D_j-modules are free, $(b_1, \cdots, b_n)e_j$ can be made the first row of a unimodular matrix $A_j \in (D_j)_{n\times n}$ (cf. §2 of [3]). Take $N_j = {}^t A_j$, and we have $(a_1, \cdots, a_n)e_j N_j = (a, a_2', \cdots, a_n')e_j$ with $a_i' \in aD_j$ for $i = 2, \cdots, n$. Then there exists a unimodular matrix N_j' such that $(a, a_2', \cdots, a_n')e_j N_j' = (a, 0, \cdots, 0)e_j$. Put $M_j = N_j N_j'$, and we have $(a_1, \cdots, a_n)e_j M_j = (a, 0, \cdots, 0)e_j$. If $ae_i = 0$, we put $M_i = e_i I_n$ where I_n is the $n\times n$ identity matrix in R. We put $M = \sum M_i$. Then M is invertible in R and $(a_1, \cdots, a_n)M = (a, 0, \cdots, 0)$.

PROOF OF THE "IF" PART OF THE THEOREM. Apply Lemma 1 in the case of $r=2$ and take the $s+1$ elements $x, y_1, \cdots, y_s$ of R. We may assume that R is indecomposable, since any direct product of OP-rings is an OP-ring. We notice that any unimodular vector is an outer product when R is an indecomposable semilocal ring (cf. §2 of [4]), since any projective module over such a ring is free (Proposition 5 of Chapter II, §5 of [1]). Now let S be the set of all ideals which are generated by the components of nonouter product vectors. We deduce a contradiction from the assumption $S \neq \varnothing$. Let $\mathfrak{a}$ be a maximal element of S. Put $\mathfrak{a} = Iv$ where $v = (a_1, \cdots, a_n) \in R^n$ is not an outer product. We have $\mathfrak{a} \neq 1$. We consider the ideal $\mathfrak{a}$ in the semilocal ring R/Rx with the maximal ideals $\mathfrak{m}_i/Rx$, $i = 1, \cdots, s$. By Lemma 2, R/Rx is a principal ideal ring, hence there exists an element a of Iv and a matrix M with coefficients in R such that $(a_1, \cdots, a_n)M \equiv (a, 0, \cdots, 0) \pmod{Rx}$ and that M is invertible mod Rx. Since $x \in J$, M is invertible in R. We put $(a_1, \cdots, a_n)M = (a', xb_2, \cdots, xb_n)$. The vectors $(a_1, \cdots, a_n)$ and $(a', xb_2, \cdots, xb_n)$ generate the same ideal $\mathfrak{a}$, since M is invertible in R. Put $w = (a', b_2, \cdots, b_n)$, $Iw = \mathfrak{b}$, and we have $\mathfrak{b} \supseteq \mathfrak{a}$. If $\mathfrak{b} \neq \mathfrak{a}$, the vector w is an outer product by virtue of the maximality of $\mathfrak{a}$, hence $(a', xb_2, \cdots, xb_n)$ and $(a_1, \cdots, a_n)$ are also outer products (cf. §2, (5) and (6) of [4]). Thus we have $\mathfrak{b} = \mathfrak{a}$, whence $a'R \subseteq \mathfrak{a} \subseteq a'R + xI(b_2, \cdots, b_n) \subseteq a'R + J\mathfrak{a}$. This implies that $\mathfrak{a} = a'R$ by Nakayama's lemma. We put $b_2 = a'c_2, \cdots, b_n = a'c_n$, $v' = (1, c_2, \cdots, c_n)$. Since v' is an outer product, $w = a'v'$ is also an outer product, hence v is an outer product. This is a contradiction.

2. This section contains some miscellaneous results related to the theorem in §1. First we give a simple proof of a result used in the proof of the theorem of Towber (cf. Case 2, p. 196 of [10]).

PROPOSITION 1. *Let R be a local ring with the maximal ideal $\mathfrak{m}$ and let $\mathfrak{p}$ be a prime ideal, $\mathfrak{p} \neq \mathfrak{m}$. Suppose that $\mu(\mathfrak{m}) \leqq 2$. Then $\mathfrak{p}$ is principal.*

PROOF (DUE TO Y. KINUGASA). Put $\mathfrak{m} = I(a, b)$. $\mathfrak{p} \neq \mathfrak{m}$ implies that $a \notin \mathfrak{p}$ or $b \notin \mathfrak{p}$, so we assume that $a \notin \mathfrak{p}$. Then the local ring $R = R/Ra$ is a principal ideal ring since $\mathfrak{m}/Ra$ is principal. Hence there exists an integer n

such that $\mathfrak{p}+Ra=Rb^n+Ra$. Put $b^n=p+ra$, $r\in R$. Take any element x of $\mathfrak{p}$. Then x can be written as $x=r'b^n+sa$ for some r', $s\in R$. Now we have $x=r'(p+ra)+sa=r'p+(rr'+s)a$. Hence we have $x-r'p=(rr'+s)a\in\mathfrak{p}$, so $rr'+s\in\mathfrak{p}$ since $a\notin\mathfrak{p}$. Therefore $x\in Rp+\mathfrak{p}a$, so we have $\mathfrak{p}=Rp+\mathfrak{p}a=Rp+\mathfrak{p}\mathfrak{m}$. Thus we have $\mathfrak{p}=Rp$ by Nakayama's lemma.

As is well known, any unimodular vector of the free module R^n generates a direct summand of R^n. The following proposition is a generalization of this fact.

PROPOSITION 2. *Let R be a ring, $M=(m_{ij})$ an $n\times s$ matrix $(n\geqq s)$ with coefficients R and let $M_1,\cdots,M_t$ $(t=\binom{n}{s})$ be the set of all $s\times s$ minors of M. Then the following conditions* (i), (ii) *and* (iii) *are equivalent*:

(i) *There exists an $s\times n$ matrix N with coefficients in R such that $NM=E_s$ $(=s\times s$ identity matrix$)$.*

(ii) $I(M_1,\cdots,M_t)=1$.

(iii) *If we put $v_j=(m_{1j},\cdots,m_{nj})$ and $D=Rv_1+\cdots+Rv_s$, D is a free direct summand of rank s of R^n.*

Most of the results of this proposition are contained in Proposition 6.1 of [**6**], but the proof is new and the context is more general.

PROOF. (i)$\Rightarrow$(ii). Let the column vectors of M be $v_1,\cdots,v_s$. Assume there exists a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{m}\supseteq I(M_1,\cdots,M_t)$. Then $v_1,\cdots,v_s$ are linearly dependent mod $\mathfrak{m}$. Hence there exists a vector $v=(x_1,\cdots,x_s)$ of R^s such that $I(x_1,\cdots,x_s)\not\subseteq\mathfrak{m}$ and $\sum x_iv_i\equiv 0 \pmod{\mathfrak{m}}$. Then we have that $M^tv\equiv 0 \pmod{\mathfrak{m}}$ and that $NM^tv=E_s{}^tv={}^tv$. This is a contradiction.

(ii)$\Rightarrow$(iii). We use the induction on s. Case of $s=1$ is Lemma 5 of [**7**], so we assume the validity of (ii)$\Rightarrow$(iii) in case of $s-1$, and prove that in case of s. Let M' be the $n\times(s-1)$ matrix obtained by deleting the last column from M, and let $M_1',\cdots,M_{t'}'$ $(t'=\binom{n}{s-1})$ be the t' minors of M'. We have, by the assumption (ii), $I(M_1',\cdots,M_{t'}')=1$. By the induction assumption, $D'=Rv_1+\cdots+Rv_{s-1}$ is a free direct summand of rank $s-1$ of R^n. We put $R^n=D'\oplus P'$, $v_s=d'+p'$, $d'\in D'$, $p'\in P'$, $p'=(r_1,\cdots,r_n)$, $r_i\in R$. We have $I(r_1,\cdots,r_n)=1$. For: let $\mathfrak{m}$ be a maximal ideal of R. $\mathfrak{m}\supseteq I(r_1,\cdots,r_n)$ implies that $\bar{D}'=(D'+\mathfrak{m}R^n)/\mathfrak{m}R^n=(D+\mathfrak{m}R^n)/\mathfrak{m}R^n$, so $\bar{D}$ is an $s-1$ dimensional vector space over the field $R/\mathfrak{m}$, but this contradicts the hypothesis $I(M_1,\cdots,M_t)=1$. Thus $I(r_1,\cdots,r_n)=1$, and this means that Rp' is a free direct summand of R^n of rank 1. Since $Rp'\subseteq P'$, Rp' is a free direct summand of P' of rank 1. If we put $P'=Rp'\oplus P''$, we have $R^n=D'\oplus P'=D'\oplus Rp'\oplus P''$, and $D=D'\oplus Rp'$ is free of rank s. This proves the implication (ii)$\Rightarrow$(iii).

Before proving the implication (iii)$\Rightarrow$(i), we write a lemma the proof of which is written for the completeness.

LEMMA. *Let M be a free R-module of rank s and $\{v_1, \cdots, v_s\}$ a generating set for M of order s, then $\{v_1, \cdots, v_s\}$ is a basis.*

PROOF. Let F be free on $\{e_1, \cdots, e_s\}$ and define an epimorphism $\eta: F \to M$ by $e_i \mapsto v_i$ for each i. Then η splits, so Ker η is projective, and rank(Ker η)$=$rank $F-$rank $M=0$.

(iii)$\Rightarrow$(i). Let $\{e'_1, \cdots, e'_s\}$ be the canonical basis of R^s, $\{e_1, \cdots, e_n\}$ the canonical basis of R^n, $R^n = D \oplus P$ and let φ be the R-linear map from R^n to R^s such that $\varphi v_i = e'_i$ for $i=1, \cdots, s$, $\varphi p = 0$ for any $p \in P$. If we put $(\varphi e_1, \cdots, \varphi e_n) = (e'_1, \cdots, e'_s)N$, we have $NM = E_s$.

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