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FACTORIZATIONS OF NONNEGATIVE MATRICES

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ABSTRACT. Suppose A is an n-square matrix over the real numbers such that all principal minors are nonzero. If A is nonnegative, then necessary and sufficient conditions are determined for A to be factored into a product $L\cdot U$, where L is a lower triangular nonnegative matrix and U is an upper triangular nonnegative matrix with $u_{ii}=1$. These conditions are given in terms of the nonnegativity of certain almost-principal minors of A.

I. **Introduction.** Suppose A is a nonnegative matrix of order n (for which we write $A\geqq 0$). The purpose of this paper is to determine necessary and sufficient conditions that A can be factored as $L\cdot U$, where L is a lower triangular nonnegative matrix, and U is an upper triangular nonnegative matrix with $u_{ii}=1$ for $i=1, 2, \cdots, n$.

We shall use the following notation. Let A be an $n\times n$ matrix over the field of real numbers. Then A_k denotes the principal submatrix of A contained in rows $1, 2, \cdots, k$. We indicate the minor of A with rows and columns indexed by $i_1, i_2, \cdots, i_p$ and $j_1, j_2, \cdots, j_p$, respectively, by $A(i_1, i_2, \cdots, i_p|j_1, j_2, \cdots, j_p)$. If A_k is nonsingular, the Schur complement of A_k in

$$A = \begin{pmatrix} A_k & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is $(A|A_k)=A_{22}-A_{21}A_k^{-1}A_{12}$ [3].

II. $A=LU$ **with** $L\geqq 0$, $U\geqq 0$.

THEOREM. *Let A be a nonnegative matrix of order n with nonzero principal minors of every order. Then the following statements are equivalent.*

(1) $A=LU$, *where L is a lower triangular nonnegative matrix, and U is an upper triangular matrix with $u_{ii}=1$.*

(2) $(A|A_k)\geqq 0$ *for* $k=1, 2, \cdots, n-1$.

(3) $A(1, \cdots, k, i|1, \cdots, k, j)\geqq 0$ *for* $k=1, \cdots, n-1$ *and* $k<i, j\leqq n$.

PROOF. We demonstrate the following implications: $1\Rightarrow 2, 2\Rightarrow 3, 3\Rightarrow 1$.

Suppose $A=LU$ where

$$A = \begin{pmatrix} A_k & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

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We partition L and U conformably with A as

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}.$$

It is immediate that $A_{22} = L_{21}U_{12} + L_{22}U_{22}$; hence $(A|A_k) = (LU|L_{11}U_{11}) = L_{21}U_{12} + L_{22}U_{22} - L_{21}U_{11}(L_{11}U_{11})^{-1}L_{11}U_{12} = L_{22}U_{22} \geqq 0$ for $k = 1, 2, \cdots, n-1$.

Next, assume that (2) holds. If $C = (A|A_k) = (c_{ij})$, then

$$c_{ij} = \frac{A(1, \cdots, k, i \mid 1, \cdots \ k, j)}{A(1, \cdots, k \mid 1, \cdots, k)} \geqq 0$$

for $k = 1, \cdots, n-1$ and $k < i, j \leqq n$ by a lemma of Crabtree and Haynsworth [**1**]. It is easily seen that

$$A(1, \cdots, k \mid 1, \cdots, k) > 0$$

for $k = 1, \cdots, n-1$. (If $k = 1$, then $a_{11} > 0$. For $k = 2$, we have

$$\frac{A(1, 2 \mid 1, 2)}{a_{11}} \geqq 0.$$

But $a_{11} > 0$, and $A(1, 2|1, 2) \neq 0$, so $A(1, 2|1, 2) > 0$. The procedure is clear for $k = 1, 2, \cdots, n-1$.) Hence

$$A(1, \cdots, k, i \mid 1, \cdots, k, j) \geqq 0$$

for $k = 1, \cdots, n-1$ and $k < i, j \leqq n$.

Finally, suppose that (3) is valid. If $n = 2$, then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & (A|a_{11}) \end{pmatrix} \begin{pmatrix} 1 & \dfrac{a_{12}}{a_{11}} \\ 0 & 1 \end{pmatrix}.$$

so a factorization exists.

Suppose the result holds for $p < n$, and assume A is of order n. Now $(A|a_{11}) = (m_{ij})$ where

$$m_{ij} = \frac{A(1, i \mid 1, j)}{a_{11}} \geqq 0$$

for $1 < i, j \leqq n$. Using a well-known identity of Sylvester [**2**, p. 101], we have

$$(A \mid a_{11})(2, \cdots, k, i \mid 2, \cdots, k, j)$$
$$= (1/a_{11})^k\{(a_{11})^{k-1}A(1, \cdots, k, i \mid 1, \cdots, k, j)\} \geqq 0$$

for $2 \leqq k \leqq n-1$ and $k<i$, $j \leqq n$. Hence, using the inductive hypothesis, there exist $L_{22} \geqq 0$ and $U_{22} \geqq 0$ with $u_{jj}=1$ such that $(A|a_{11})=L_{22}U_{22}$. Partition

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{22} is of order $n-1$. Let $l_{11}=a_{11}$, $L_{21}=A_{21}$, and $U_{12}=A_{12}/a_{11}$. Then $A_{22}=(1/a_{11})A_{21}A_{12}+L_{22}U_{22}=L_{21}U_{12}+L_{22}U_{22}$. Now if

$$L = \begin{pmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & U_{12} \\ 0 & U_{22} \end{pmatrix},$$

we have $A=LU$ where $L \geqq 0$, $U \geqq 0$ and $u_{ii}=1$.

The proof of the theorem is now complete.

We would like to observe that there is no loss of generality in (1) in assuming that the diagonal of U consists entirely of 1's. If A has a factorization LU where $L \geqq 0$, $U \geqq 0$, then let $D=\operatorname{diag}(u_{11}, u_{22}, \cdots, u_{nn})$. Then $A=(LD)(D^{-1}U)=\hat{L}\hat{U}$, where $\hat{L} \geqq 0$, $\hat{U} \geqq 0$ and $\hat{U}$ has all 1's on the main diagonal.

The theorem could be stated in an alternative form involving the Schur complement, i.e. A is factorable as LU with $L \geqq 0$, $U \geqq 0$ if and only if $(A|a_{11})$ is factorable as $L'U'$ where $L' \geqq 0$, $U' \geqq 0$ with $u_{ii}=1$. Moreover, it is clear that similar results can be obtained for factorizations of the type $U \cdot L$, and these results are independent of each other. For example, if

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{pmatrix},$$

then $A=LU$. But $A(1, 3|2, 3)<0$, so there is no nonegative factorization of the type UL.

References

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