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RINGS SATISFYING MONOMIAL IDENTITIES

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ABSTRACT. The following theorem is proved: Suppose R is an associative ring and suppose that $w(x_1, \cdots, x_n)$ is a fixed word distinct from $x_1 \cdots x_n$. If, further, $x_1 \cdots x_n = w(x_1, \cdots, x_n)$, for all $x_1, \cdots, x_n$ in R, then the commutator ideal of R is nilpotent. Moreover, it is shown that this theorem need not be true if the word w is not fixed.

Suppose R is an associative ring and suppose $x_1, \cdots, x_n$ are elements of R. A *word* $w(x_1, \cdots, x_n)$ in $x_1, \cdots, x_n$ is a product in which each factor is x_i for some $i=1, \cdots, n$. Our present object is to prove

THEOREM 1. *Suppose R is an associative ring and suppose $w(x_1, \cdots, x_n)$ is a fixed word distinct from the word $x_1 \cdots x_n$. Suppose*

(1) $$x_1 \cdots x_n = w(x_1, \cdots, x_n), \quad \textit{for all } x_1, \cdots, x_n \textit{ in } R.$$

Then there exists a positive integer m such that $R^m C(R) R^m = (0)$, where $C(R)$ is the commutator ideal of R. In particular, the commutator ideal of R is nilpotent.

Moreover, a counterexample is given which shows that Theorem 1 need not be true if $w(x_1, \cdots, x_n)$ is not a fixed word.

In preparation for the proof of Theorem 1, we first show the following lemmas.

LEMMA 1. *Suppose R is an associative semisimple ring, and suppose $w(x_1, \cdots, x_n)$ is a fixed word involving each of the elements $x_1, \cdots, x_n$ of R. If, further,*

(2) $$x_1 \cdots x_n = w(x_1, \cdots, x_n), \quad \textit{for all } x_1, \cdots, x_n \textit{ in } R,$$

then R is commutative.

PROOF. Suppose, first, that R has an identity 1. We now distinguish two cases.

Case 1.

(3) $$x_1 \cdots x_n = w(x_1, \cdots, x_n) = x_{\sigma(1)} \cdots x_{\sigma(n)},$$

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where σ is a permutation of $\{1, \cdots, n\}$ distinct from the identity permutation. Then, for some integers i, j, we have $i<j$ but $\sigma(i)>\sigma(j)$. Now, let $a, b \in R$, and set in (3), $x_{\sigma(i)}=a$, $x_{\sigma(j)}=b$, $x_k=1$ for all $k \neq \sigma(i), k \neq \sigma(j)$, we get $ba=ab$, and the lemma follows.

Case 2.

$$x_1 \cdots x_n = w(x_1, \cdots, x_n), \quad \text{some } x_t \text{ appears at least twice in } w(x_1, \cdots, x_n). \tag{4}$$

In this case, by setting $x_1=\cdots=x_{t-1}=x_{t+1}=\cdots=x_n=1$ in (4), we get

$$x_t = x_t^k, \quad \text{for all } x_t \text{ in } R \qquad (k > 1). \tag{5}$$

Hence [**2**, p. 217], R is commutative, and the lemma follows again.

Returning to the general case, observe that, since R is semisimple, R is isomorphic to a subdirect sum of primitive rings R_i, $i \in \Gamma$, each of which clearly satisfies (2). Since every subring and every homomorphic image of R satisfies (2), it follows [**2**, p. 33] that some complete matrix ring, Δ_m, over a division ring satisfies (2) also. Since Δ_m has an identity, it follows (by the first part of this proof) that Δ_m is commutative. Thus $m=1$, and $\Delta_m=\Delta$ is a field. Hence [**2**, p. 33] the primitive ring R_i is isomorphic to the field Δ. Thus R is isomorphic to a subdirect sum of fields, and hence R is commutative. This proves the lemma.

Next, we consider the case in which the word $w(x_1, \cdots, x_n)$ satisfies (4). In this case, we can even say more. Indeed, we have

LEMMA 2. *Suppose R is an associative ring and suppose that $C(R)$ and J denote the commutator ideal and Jacobson ideal of R. Suppose that $w(x_1, \cdots, x_n)$ is a fixed word involving each of the elements $x_1, \cdots, x_n$ of R. Suppose, moreover, that for some t, x_t appears at least twice in $w(x_1, \cdots, x_n)$. If, further,*

$$x_1 \cdots x_n = w(x_1, \cdots, x_n), \quad \textit{for all } x_1, \cdots, x_n \textit{ in } R, \tag{6}$$

then (i) *R/J is isomorphic to a subdirect sum of finite fields of orders bounded by the length of w*; (ii) $C(R) \subseteq J$; (iii) *J consists of precisely the set of nilpotent elements of R.*

PROOF. Since R/J is a semisimple ring which, clearly, satisfies (6), it follows, by Lemma 1, that R/J is commutative, and hence R/J is isomorphic to a subdirect sum of fields F_i, $i \in \Gamma$. Now, each F_i clearly satisfies (6), and hence by setting $x_i=1$ for all $i \neq t$ in (6), we obtain

$$x_t = x_t^k, \quad \text{for all } x_t \text{ in } R \qquad (k > 1). \tag{7}$$

Therefore F_i is a finite field with at most k elements, and, clearly, k is equal to or less than the length of the word $w(x_1, \cdots, x_n)$. This proves (i).

Part (ii) follows at once, since R/J is commutative. Finally, to prove (iii), suppse $a \in J$, and set $x_i = a$, for all i, in (6). We get, $a^n = a^n a^l$ for some $l \geqq 1$, and hence $a^n = 0$. Conversely, if a is nilpotent, then $\bar{a}$ $(=a+J)$ is a nilpotent element in R/J, and hence by (i), $\bar{a} = \bar{0}$. Thus $a \in J$, and the lemma is proved.

Next, we prove

LEMMA 3. *Suppose R is an associative ring, and suppose J is the Jacobson radical of R. Suppose that $w(x_1, \cdots, x_n)$ is a fixed word involving each of the elements $x_1, \cdots, x_n$ of R and in which some x_t appears at least twice. Suppose, moreover, that*

$$(8) \qquad x_1 \cdots x_n = w(x_1, \cdots, x_n), \quad \textit{for all } x_1, \cdots, x_n \textit{ in } R.$$

Then, for some i, $1 \leqq i \leqq n$, we have $R^{i-1} J R^{n-i} = (0)$.

PROOF. By Lemma 2 (iii), J is a nil ring. Now, let $a \in J$, and set in (8), $x_1 = \cdots = x_n = a$, we get $a^n = a^n a^l$, for some $l \geqq 1$. Therefore the nil ring J satisfies $a^n = 0$, and thus [**1**, p. 28] J is locally nilpotent. Next, let $a_1, \cdots, a_n \in J$. Then the ring generated by $a_1, \cdots, a_n$ is nilpotent, say of index k. Now, by reiterating (8) until the length of the word $w(x_1, \cdots, x_n)$ in the right-hand side becomes $\geqq k$, it follows that $a_1 \cdots a_n = 0$, and hence $J^n = (0)$. Next, since x_i appears at least twice in the word $w(x_1, \cdots, x_n)$, we can, by reiterating in (8), obtain a word $w'(x_1, \cdots, x_n)$ of length $\geqq n^2$ and such that

$$(9) \qquad x_1 \cdots x_n = w'(x_1, \cdots, x_n), \quad \text{for all } x_1, \cdots, x_n \text{ in } R.$$

Observe that in the word $w'(x_1, \cdots, x_n)$, some x_i appears at least n times. We now fix i, and substitute $x_i = a$; $x_j = r_j$, $j \neq i$, where each $r_j \in R$, we get

$$r_1 \cdots r_{i-1} a r_{i+1} \cdots r_n \in J^n = (0).$$

Hence, $R^{i-1} J R^{n-i} = (0)$, and the lemma is proved.

Our final lemma is true for semigroups (and hence, *a fortiori*, for rings), and has been proved in [**4**, Theorem 1].

LEMMA 4. *Let S be a semigroup such that, for all $x_1, \cdots, x_n$ in S,*

$$x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where σ is a fixed permutation of $\{1, \cdots, n\}$ distinct from the identity permutation. Then there exists an integer m such that, for all u, v in S^m and all x, y in S, we have $uxyv = uyxv$.

We are now in a position to prove Theorem 1.

PROOF OF THEOREM 1. First, suppose the word $w(x_1, \cdots, x_n)$ does *not* involve x_i, for some i. In (1), set $x_i=0$ and, for $j\neq i$, let x_j be arbitrary; we get $w(x_1, \cdots, x_n)\equiv 0$ and hence, by (1), $x_1 \cdots x_n=0$ for *all* $x_1, \cdots, x_n$ in R (since $w(x_1, \cdots, x_n)$ is fixed). Thus $R^n=(0)$, and Theorem 1 follows at once. Next, suppose $w(x_1, \cdots, x_n)=x_{\sigma(1)} \cdots x_{\sigma(n)}$, for some permutation σ of $\{1, \cdots, n\}$ different from the identity. Then, by Lemma 4,

$$u(xy - yx)v = 0, \quad \text{for all } u, v \in R^m \text{ and all } x, y \in R.$$

Hence, $R^mC(R)R^m=(0)$, and Theorem 1 follows again. The only case left is when $w(x_1, \cdots, x_n)$ involves each x_i and, moreover, some x_t appears at least twice in $w(x_1, \cdots, x_n)$. By Lemmas 2 and 3 we have $R^{i-1}C(R)R^{n-i}\subseteq R^{i-1}JR^{n-i}=(0)$, for some i, $1\leqq i\leqq n$, and once again the theorem follows. This completes the proof.

COROLLARY. *Suppose R is an associative semiprime ring satisfying the hypothesis of Theorem* 1. *Then R is commutative.*

PROOF. Since R is a semiprime ring, the prime radical of R is (0) [**3**, p. 146], and hence R contains no nonzero nilpotent ideals. Now, by Theorem 1, the commutator ideal, $C(R)$, of R is nilpotent, and hence $C(R)=(0)$. Therefore R is commutative, and the corollary is proved.

We conclude with the following

REMARK. Theorem 1 need not be true if we replace the fixed word $w(x_1, \cdots, x_n)$ by a "variable" word (depending on $x_1, \cdots, x_n$). For, suppose R is the complete matrix ring, $(GF(2))_2$, of all 2×2 matrices over $GF(2)$. It is easily verified that

$$\begin{aligned} x_1x_2 &= x_1^7x_2 && \text{if } x_1 \text{ is invertible or idempotent,} \\ &= x_1x_2^7 && \text{if } x_2 \text{ is invertible or idempotent,} \\ &= (x_1x_2)^2 && \text{otherwise.} \end{aligned} \tag{10}$$

However, the commutator ideal of $(GF(2))_2$ is not even nil. In verifying (10), observe that (i) $x^8=x^2$ holds in $(GF(2))_2$; (ii) every matrix in $(GF(2))_2$ is invertible, or idempotent, or nilpotent; (iii) the product of any two nilpotent matrices in $(GF(2))_2$ is idempotent.

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REFERENCES

1. I. N. Herstein, *Theory of rings*, Math. Lecture Notes, University of Chicago, Chicago, Ill., 1961.

2. N. Jacobson, *Structure of rings*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R.I., 1964. MR **36** #5158.

3. N. H. McCoy, *The theory of rings*, Macmillan, New York; Collier-Macmillan, London, 1964. MR **32** #5680.

4. M. S. Putcha and A. Yaqub, *Semigroups satisfying permutation identities*, Semigroup Forum **13** (1971), 68–73.

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