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ON THE FRATTINI SUBGROUP OF THE GENERALIZED FREE PRODUCT WITH AMALGAMATION

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ABSTRACT. Let G be the generalized free product of two groups A and B amalgamating their subgroup H. If H satisfies the minimal condition on subgroups then the Frattini subgroup of G is contained in H. When H is finite then this Frattini subgroup can be precisely determined.

1. **Notation.** Our notation is standard: $H \leqq G$ means H is a subgroup of G; $H < G$ is used for proper subgroups; $H \lhd G$ for normal subgroups.

By $G=(A*B)_H$ we denote the generalized free product of A and B amalgamating their common subgroup H.

If $x \in G \backslash H$ then x has exactly one of the following forms:

(1) $x=a_1b_1a_2b_2 \cdots a_m$,

(2) $x=a_1b_1a_2b_2 \cdots a_mb_m$,

(3) $x=b_1a_1b_2a_2 \cdots b_ma_m$,

(4) $x=b_1a_1b_2a_2 \cdots b_m$,

where m is a positive integer and $a_i \in A \backslash H$, $b_i \in B \backslash H$.

We shall write $x \in {}_AW$ if x has the form (1) or (2). Similarly $x \in {}_BW$ if x has the form (3) or (4). Also, $x \in W_A$ if x has form (1) or (3) and $x \in W_B$ if x has form (2) or (4).

By $\Phi(G)$ we denote the Frattini subgroup of a group G.

2. **Main results.** By an easy application of Zorn's lemma one can show that if $H \leqq G$ there exists a unique maximal normal subgroup of G contained in H. More precisely this subgroup, say $K(H, G)$, is characterized by the following properties:

(i) $K(H, G) \lhd G$,

(ii) $K(H, G) \leqq H$,

(iii) $X \lhd G \,\&\, X \leqq H \Rightarrow X \leqq K(H, G)$.

It is immediate that for $x \in G$ the following holds:

(5) $x \in K(H, G) \Leftrightarrow (\forall y \in G) y^{-1}xy \in H$.

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The following lemma is due to A. Whittemore [**2**, Proposition 2.3].

LEMMA 1. *If* $G=(A*B)_H$ *and there exists* $x\in G$ *such that* $H\cap xHx^{-1}=1$ *then* $\Phi(G)=1$.

Our main result is

THEOREM 1. *If* $G=(A*B)_H$, $K(H, G)=1$, *and* H *satisfies the minimum condition on subgroups then* $\Phi(G)=1$.

PROOF. Graham Higman and B. H. Neumann have shown [**1**] that $H=1$ implies $\Phi(G)=1$. Thus we shall assume that $H\neq 1$.

Consider the set of all subgroups of G of the form $H\cap xHx^{-1}$, $x\in G$. By our minimal condition there exists a minimal element in this set, say $M=H\cap uHu^{-1}$. Since $K(H, G)=1$ and $H\neq 1$ it follows that M is a proper subgroup of H and consequently $u\notin H$, say $u\in W_A$.

Assume that $M\neq 1$ and choose $y\in M$, $y\neq 1$. Then $u^{-1}yu\in H$. It will be shown in Lemma 2 that there exists $v\in{}_BW$ such that $v^{-1}u^{-1}yuv\notin H$.

If $h\in H$ then

$$h \notin M \Leftrightarrow u^{-1}hu \notin H \Rightarrow v^{-1}u^{-1}huv \notin H \Leftrightarrow h \notin (uv)H(uv)^{-1}.$$

Hence $H\cap(uv)H(uv)^{-1}$ is a subgroup of M. It is a proper subgroup of M because $y\in M$ and $y\notin(uv)H(uv)^{-1}$. This contradicts our choice of M. Therefore we must have $M=1$ and the assertion follows from Lemma 1.

It remains to prove the next lemma which asserts a little bit more than we used in the proof of Theorem 1.

LEMMA 2. *Let* $G=(A*B)_H$, $K(H, G)=1$ *and assume that* H *satisfies the minimum condition on subgroups normal in* B. *Then for every* $h\in H$, $h\neq 1$, *there exists* $w\in{}_BW$ *such that* $w^{-1}hw\notin H$.

PROOF. Let $H=H_0$ and define

$$H_{2i+1} = K(H_{2i}, A), \qquad i \geqq 0,$$
$$H_{2k} = K(H_{2k-1}, B), \qquad k \geqq 1.$$

We have $H_0\geqq H_1\geqq H_2\geqq \cdots$, and $\bigcap_{i\geqq 0} H_i=1$ because $K(H, G)=1$. Our minimum condition implies that $H_n=1$ for large n. Let m be the smallest integer such that $H_m=1$. Then we have

$$H_0 \geqq H_1 > H_2 > \cdots > H_m = 1. \tag{6}$$

Let j be the unique integer such that $h\in H_j\backslash H_{j+1}$. We shall distinguish two cases:

Case 1. $j=2k-1$ is odd. Then by (5) there exists $b_1\in B\backslash H$ such that $b_1^{-1}hb_1\in H_{2k-2}\backslash H_{2k-1}$. Now, there exists $a_1\in A\backslash H$ such that $a_1^{-1}b_1^{-1}hb_1a_1\in$

$H_{2k-3}\backslash H_{2k-2}$. Continuing in this manner we get $w=b_1a_1 \cdots \in_B W$ with the property that $w^{-1}hw \notin H$.

Case 2. $j=2i$ is even. Take $b_1 \in B\backslash H$ and consider $b_1^{-1}hb_1$. If $b_1^{-1}hb_1 \in H_{2i}\backslash H_{2i+1}$ then we can continue as in Case 1.

Suppose that $b_1^{-1}hb_1 \in H_{2i+1}\backslash H_{2i+2}$. Then we take $a_1 \in A\backslash H$ and consider $a_1^{-1}b_1^{-1}hb_1a_1$. If $a_1^{-1}b_1^{-1}hb_1a_1 \in H_{2i+1}\backslash H_{2i+2}$ we can continue as in Case 1. If $a_1^{-1}b_1^{-1}hb_1a_1 \in H_{2i+2}\backslash H_{2i+3}$ then we transform this element by $b_2 \in B\backslash H$. It is clear that because the sequence (6) is finite the Case 2 after a finite number of such steps must reduce to Case 1.

The lemma is proved.

It follows from Theorem 1 that if $G=(A*B)_H$, where H is finite, then $\Phi(G) \leqq K(H, G)$.

References

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