SUBMANIFOLDS OF EUCLIDEAN SPACE WITH PARALLEL SECOND FUNDAMENTAL FORM

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In this paper necessary conditions are given for a complete Riemannian manifold M^n to admit an isometric immersion into R^{n+p} with parallel second fundamental form. Namely, it is shown that M^n must be affinely equivalent either to a totally geodesic submanifold of the Grassmann manifold $G(n, p)$, or to a fibre bundle over such a submanifold, with Euclidean space as fibre and the structure being close to a product. (An affine equivalence is a diffeomorphism that preserves Riemannian connections.) The proof depends on the auxiliary result that the second fundamental form is parallel iff the Gauss map is a totally geodesic map.

1. **The main result.** Let $i:M^n \to R^{n+p}$ be an immersion of a manifold M^n into Euclidean space, and endow M with the induced Riemannian metric. The second fundamental form α of the immersion i is a symmetric section of the bundle of bilinear maps $L^2(TM, TM; NM)$, where TM and NM denote the tangent and normal bundles of M, respectively [4]. We identify this bundle of maps with the isomorphic bundle $L(TM, L(TM, NM))$, and α with the corresponding section. At each point of M, the kernel of α (in the second interpretation) is called the relative nullity space, and its dimension k, the relative nullity index [2]. We say the second fundamental form α is parallel if $D\alpha = 0$ (D always denotes the appropriate covariant derivative). Note α parallel implies k constant, whence it follows that the relative nullity spaces are tangent to a foliation of M whose leaves are flat totally geodesic submanifolds and are portions of affine subspaces of R^{n+p}. Let $G(n, p)$ denote the Grassmann manifold of n-planes through the origin in R^{n+p}, endowed with its natural Riemannian symmetric space metric. Finally, let $E(k)$ denote the group of Euclidean motions in R^k. Then our main result is the following:

THEOREM. *Suppose α is parallel and M is complete. Then*

(i) *if $k=0$, M is a complete totally geodesic submanifold of $G(n, p)$, or*

(ii) *if $k \geq 1$, there is an $(E(k), R^k)$-fibration $M \to B$, where B is a complete totally geodesic submanifold of $G(n, p)$ and the fibres are the leaves of the relative nullity foliation. The metric of M is composed from those on base and fibre, and the fibration admits an integrable connection with totally geodesic horizontal leaves (i.e. it is a totally geodesic Riemannian submersion, see [7]).*

(iii) *The original Riemannian connection of M, or its projection onto B, respectively, coincides with the connection induced from $G(n, p)$.*

(iv) *M has nonnegative curvature, and is locally symmetric.*

We note two related results. First, Riemannian globally symmetric spaces are known to be diffeomorphic to products of form $V \times R^q$, where V is a compact globally symmetric space [5, p. 172]. Second, arbitrary complete Riemannian manifolds of nonnegative curvature have been shown to be diffeomorphic to vector bundles over a compact totally geodesic submanifold [1].

2. Auxiliary results and idea of proof.   Next we state the lemmas used in the proof of our theorem. They have independent interest, since they do not depend on the special assumption that α is parallel. Let $g: M \to G(n, p)$ denote the Gauss map of the immersion i; namely, for each x in M, $g(x)$ is defined to be the n-plane in R^{n+p} tangent to $i(M)$ at $i(x)$, translated to the origin. By pulling the bundle $TG(n, p)$ back over M via g, we may consider the differential of g as a map $g_*: TM \to g^{-1}TG(n, p)$. Call a map of Riemannian manifolds totally geodesic if it maps geodesics into geodesics and preserves the affine parameter [7]. Then an immersion is a totally geodesic immersed submanifold iff as a map it is totally geodesic with the induced metric.

LEMMA 1 [6, §2].   *There exists an isomorphism $F: L(TM, NM) \cong g^{-1}TG(n, p)$ satisfying $g_* = F \circ \alpha$.*

LEMMA 2.   *F preserves connections, i.e. $Dg_* = F \circ D\alpha$.*

LEMMA 3.   *The map g is totally geodesic iff α is parallel.*

LEMMA 4.   *An immersion $f: M \to M'$ of Riemannian manifolds is a totally geodesic map iff M is a totally geodesic immersed submanifold of M' and the original connection of M coincides with the connection induced by f.*

The theorem of §1 is proved by applying to Lemma 3 our previous results [7] on the global structure of totally geodesic maps with complete domain, together with Lemma 4 which complements those results.

3. **Proof of Lemma 2.** We translate the map F of Lemma 1 from a vector bundle context to a principal bundle one and apply [**4**, p. 8, Theorem 2.2]. The general scheme is the following. Suppose $P \to M$ and $P' \to M'$ are principal K-bundles and $f: P \to P'$ is a bundle map over the map $g: M \to M'$. If $U \to M$ and $U' \to M'$ are associated (K, W)-bundles, then f induces a K-isomorphism $F: U \to U'$ as follows. For each x in M, we have $F_x = (fz)z^{-1}: U_x \to W \to U'_{gx}$, where z is an arbitrary element of P_x, viewed as an isomorphism $W \to U_x$ (subscripts denote fibres). If f is connection preserving, the same must be true for F, with respect to the associated connections.

Now for the situation of Lemma 1, the principal bundles are the bundle of adapted frames $P \to M$ and the bundle $E = O(n+p) \to G(n, p)$ [**4**, Chapter 7, §2]. The group for both is $O(n) \times O(p)$, and the fibre of the associated bundles is $L(R^n, R^p)$. The connections involved are the associated ones. Each element z of P_x is a pair (e, h) of isometries $e: R^n \to M_x$, $h: R^p \to N_x$, where M_x, N_x denote fibres of TM, NM, respectively, considered as subspaces of R^{n+p}. The map z^{-1} is the isomorphism $u \mapsto h^{-1}ue: L(M_x, N_x) \to L(R^n, R^p)$. If we let f be the connection preserving bundle map $P \to E$ of [**4**, p. 8, Theorem 2.2], then fz is the isometry of R^{n+p} defined by (e, h). Considered as an isomorphism $L(R^n, R^p) \to T_{gx}G(n, p)$, fz is the composition of isometries

$$L(R^n, R^p) \xrightarrow{\;h\circ\;} L(R^n, N_x) \xrightarrow{\;\pi_*\;} T_{gx}G,$$

where $\pi: V(n, p) \to G(n, p)$ is the Stiefel fibration. (The map π_* is an isometry since $L(R^n, N_x)$ is precisely the horizontal space in $TV(n, p)$ at e in $V(n, p)$.) Thus $(fz)z^{-1}: L(M_x, N_x) \to T_{gx}G$ is the composition

$$L(M_x, N_x) \xrightarrow{\;\circ e\;} L(R^n, N_x) \xrightarrow{\;\pi_*\;} T_{gx}G,$$

which is precisely the map F_x of Lemma 1. Hence F is connection preserving.

4. **Proofs of Lemmas 3 and 4.** A map $f: M \to M'$ of Riemannian manifolds is totally geodesic iff its fundamental form, $\beta(f)$, vanishes (see [7] and [3, p. 123]). But $\beta(f)$ is by definition Df_*, where f_* is considered as a map $f_*: TM \to f^{-1}TM'$, i.e. a section of the bundle of linear maps $L(TM, f^{-1}TM')$. Hence Lemma 3 follows immediately from Lemma 2.

Now suppose f is an immersion, let h, h' denote the original metrics on M, M' respectively, and let $h'' = f^{-1}h'$ denote the induced metric on M. Consider the following factorization of f,

$$(M, h) \xrightarrow{\;\text{id}\;} (M, h'') \xrightarrow{\;f\;} (M', h').$$

The composition formula for $\beta(f)$ [3, p. 131] gives

$$\beta(f) = \beta'(f) + f_* \circ \beta(\mathrm{id}),$$

where the prime on β means h'' is used on M. Now $\beta'(f)$ is normal to $f_* TM$ in TM', because it is the fundamental form of f with respect to the metric induced by f [3, p. 118]; and clearly $f_* \circ \beta(\mathrm{id})$ is in $f_* TM$. Hence $\beta(f)=0$ iff both of these quantities vanish, i.e. f is totally geodesic for the induced metric h'' and id preserves the connections of h and h''. This proves Lemma 4.

5. **Proof of the theorem.** By Lemma 3, g is totally geodesic, and by Lemma 1 the relative nullity index k is equal to the dimension of kernel g_* and the relative nullity foliation of M coincides with the kernel g_* foliation (see [7, Lemma 2.1]). Thus in case $k=0$, the map $g: M \to G(n, p)$ is a totally geodesic immersion of Riemannian manifolds. Lemma 4 then proves statement (i) and part of (iii), if one notes that immersed totally geodesic submanifolds of $G(n, p)$ are already imbedded, due to rigidity [4, Chapter 11, §4].

In case $k \geq 1$, we may apply [7, Theorem 2.2] since M is assumed complete. We see g factors into a totally geodesic Riemannian submersion $\pi: M \to B$ followed by a totally geodesic immersion $f: B \to G(n, p)$. The fibres of π are just the leaves of the k-dimensional kernel g_* foliation, whence B is a Riemannian manifold of dimension $n-k$, and is complete by [7, Theorem 3.6]. Lemma 4 and the remark made above for M in case $k=0$ give the parts of (ii) and (iii) referring to B. The same theorem [7, Theorem 3.6] also yields directly the rest of (ii), except for the fibration being an $(E(k), R^k)$-fibration. It says merely that the group is the group of isometries of a fibre. The fibres, as noted in §1 above the statement of the theorem, are flat (connected) submanifolds of M consisting of portions of affine subspaces of R^{n+p}. Since they are closed, they must be complete, whence connectivity implies they are whole affine subspaces of R^{n+p}. But then their flatness (in the metric induced from M) implies they are isometric to R^k with its standard metric. Hence the structure group is also the Euclidean group $E(k)$. This finishes the proof of (ii).

To show M has nonnegative curvature, observe that a complete totally geodesic submanifold of $G(n, p)$ is a compact globally symmetric space, and so has nonnegative curvature. This takes care of the case $k=0$. If $k \geq 1$, then the statement in (ii) about the metric of M implies that locally M has the product metric of the base and the fibre, having curvature nonnegative and zero, respectively. But then the curvature of M must itself be nonnegative.

The local symmetry of M follows directly from the assumption $D\alpha=0$,

via the Gauss equation for the curvature tensor of M. This finishes the proof.

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