

UNIFORM BOUNDEDNESS IN METRIC SPACES¹

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ABSTRACT. Let (X, d) , (Y, d') be metric spaces, \mathcal{F} a family of maps from X into Y . Let $x_0 \in X$. \mathcal{F} is said to be pointwise-bounded if for each $x \in X$, $\sup\{d'(Tx, Tx_0) \mid T \in \mathcal{F}\}$ is finite, and is uniformly bounded if for each $p > 0$, $\sup\{d'(Tx, Tx_0) \mid T \in \mathcal{F}, d(x, x_0) \leq p\}$ is finite. The major result of this paper is to place a sufficient condition on the maps in \mathcal{F} to ensure that, if X is complete, a pointwise-bounded family of continuous maps is uniformly bounded, and to show that this result is best possible.

The object of this paper is to present a general theorem in which uniform boundedness is a consequence of pointwise-boundedness. The technique used is the abstract nonlinear form of the gliding hump method first employed by Lebesgue and Toeplitz. An attractive linear application of this argument can be found in the paper of V. Ptak referred to in the bibliography.

The following definition, though somewhat awkward in appearance, is crucial to the development of the theory.

DEFINITION 1. Let (X, d) , (Y, d') be metric spaces, $T: X \rightarrow Y$. Let $x, y \in X$ be such that $Tx \neq Ty$. If $0 < \alpha < 1$, define

$$\begin{aligned} \psi(T, x, y; \alpha) \\ = \sup\{M \mid \exists z \in X \text{ with } d(x, z) \leq \alpha d(x, y) \text{ and } Md'(Tx, Ty) \leq d'(Tx, Tz)\}. \end{aligned}$$

If there is no such z , define $\psi(T, x, y; \alpha) = 0$.

Note that $\psi(T, x, y; \alpha) \geq 0$, and if $0 \leq M < \psi(T, x, y; \alpha)$, then there is a $z \in X$ with $d(x, z) \leq \alpha d(x, y)$ and $d'(Tx, Tz) \geq Md'(Tx, Ty)$.

DEFINITION 2. Let (X, d) , (Y, d') be metric spaces, \mathcal{F} a family of maps from X into Y . If $x, y \in X$, define $\psi(x, y; \alpha) = \inf\{\psi(T, x, y; \alpha) \mid T \in \mathcal{F}, Tx \neq Ty\}$ for $0 < \alpha < 1$. Finally, if $0 < \alpha < 1$, let $\kappa(\alpha) = \inf\{\psi(x, y; \alpha) \mid x \neq y\}$.

We come now to the major result of this paper.

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THEOREM 1. *Let X be a complete metric space, Y a metric space. Let \mathcal{F} be a pointwise-bounded family of continuous maps from X into Y , and let $\kappa(\alpha) > 0$ for each $\alpha \in (0, 1)$. Then \mathcal{F} is uniformly bounded.*

PROOF. Let $x_0 \in X$, and let $p < 0$. Assume that the theorem is false. We then assert that we can prove the following: given $T_1, \dots, T_n \in \mathcal{F}$, $x_1, \dots, x_n \in X$ with $d(x_0, x_k) < p$ ($1 \leq k \leq n$), and $M > 0$, then we can find an $x_{n+1} \in X$ and $T_{n+1} \in \mathcal{F}$ with $d(x_0, x_{n+1}) < p$, $d(x_n, x_{n+1}) < 1/2^{n+1}$,

$$d'(T_{n+1}x_{n+1}, T_{n+1}x_0) > M, \text{ and } d'(T_kx_n, T_kx_{n+1}) < 1/2^{n+1}$$

for $1 \leq k \leq n$. To prove this assertion, let $S(x) = \sup\{d'(Tx, Tx_0) | T \in \mathcal{F}\}$, and assume $M > 0$, $x_1, \dots, x_n \in X$, and $T_1, \dots, T_n \in \mathcal{F}$ are given as above. If the theorem is false, \mathcal{F} is not uniformly bounded on the sphere of radius p and center x_0 , so given $K > 0$ there is a $z \in X$ and $T \in \mathcal{F}$ with $d(x_0, z) < p$ and $K < d'(Tx_0, Tz)$. Consequently

$$K < d'(Tx_0, Tz) \leq d'(Tx_0, Tx_n) + d'(Tx_n, Tz) \leq S(x_n) + d'(Tx_n, Tz).$$

So we can always find a $T \in \mathcal{F}$, $z \in X$ with $d(x_0, z) < p$ and $d'(Tx_n, Tz)$ arbitrarily large.

Since T_1, \dots, T_n are continuous at x_n , choose δ_k ($1 \leq k \leq n$) such that $x \in X$, $d(x, x_n) < \delta_k \Rightarrow d'(T_kx, T_kx_n) < 1/2^{n+1}$. Now define

$$\alpha = \frac{1}{3} \min(1/2^{n+1}p, \delta_1/2p, \dots, \delta_n/2p, (p - d(x_0, x_n))/p),$$

then $0 < \alpha < 1$. Define $\mu(\alpha) = \frac{1}{2}\kappa(\alpha)$ if $\kappa(\alpha)$ is finite, and $\mu(\alpha) = 1$ otherwise. Now choose $y \in X$, $T (= T_{n+1}) \in \mathcal{F}$ with $d(x_0, y) < p$, $\mu(\alpha)d'(Tx_n, Ty) > M + S(x_n)$. Since $\mu(\alpha) < \kappa(\alpha)$, choose $x_{n+1} \in X$ with $d(x_n, x_{n+1}) \leq \alpha d(x_n, y)$ and $d'(Tx_n, Tx_{n+1}) \geq \mu(\alpha)d'(Tx_n, Ty)$. We now make the following observations:

- (1) $d(x_n, x_{n+1}) \leq \alpha d(x_n, y) \leq 2p\alpha < 1/2^{n+1}$.
- (2) if $1 \leq k \leq n$, $d'(T_kx_n, T_kx_{n+1}) < 1/2^{n+1}$, since $d(x_n, x_{n+1}) \leq 2p\alpha < \delta_k$.
- (3) $d(x_0, x_{n+1}) \leq d(x_0, x_n) + d(x_n, x_{n+1}) \leq d(x_0, x_n) + 2p\alpha < p$.
- (4) $\mu(\alpha)d'(Tx_n, Ty) \leq d'(Tx_n, Tx_{n+1}) \leq d'(Tx_n, Tx_0) + d'(Tx_0, Tx_{n+1}) \leq S(x_n) + d'(Tx_0, Tx_{n+1}) \Rightarrow d'(Tx_0, Tx_{n+1}) \geq \mu(\alpha)d'(Tx_n, Ty) - S(x_n) > M$.

This completes the proof of the assertion.

We now use the assertion just established to set up an induction. Since \mathcal{F} is not uniformly bounded, we can find $x_1 \in X$, $T_1 \in \mathcal{F}$ with $d'(T_1x_0, T_1x_1) > 2$. Having chosen $x_1, \dots, x_n \in X$, with $d(x_0, x_k) < p$ for $1 \leq k \leq n$, and $T_1, \dots, T_n \in \mathcal{F}$, choose $x_{n+1} \in X$, $T_{n+1} \in \mathcal{F}$ with $d(x_0, x_{n+1}) < p$, $d(x_n, x_{n+1}) < 1/2^{n+1}$, $d'(T_{n+1}x_0, T_{n+1}x_{n+1}) > n+2$, and $d'(T_kx_n, T_kx_{n+1}) < 1/2^{n+1}$ for $1 \leq k \leq n$. Clearly $\{x_n | n=1, 2, \dots\}$ is Cauchy; by completeness there is an $x \in X$ with $x_n \rightarrow x$. Clearly $d(x_0, x) \leq p$. Since T_n is continuous, we see that

$$\lim_{m \rightarrow \infty} d'(T_nx, T_nx_{m+1}) = 0.$$

So

$$d'(T_n x_0, T_n x_n) \leq d'(T_n x_0, T_n x) + d'(T_n x, T_n x_n)$$

$$\leq \dots \leq d'(T_n x_0, T_n x) + \sum_{k=n}^m d'(T_n x_k, T_n x_{k+1}) + d'(T_n x, T_n x_{m+1});$$

as $m \rightarrow \infty$ we obtain

$$d'(T_n x_0, T_n x_n) \leq d'(T_n x_0, T_n x) + \sum_{k=n}^{\infty} d'(T_n x_k, T_n x_{k+1})$$

$$< d'(T_n x_0, T_n x) + \sum_{k=n}^{\infty} 2^{-k-1}$$

$$< d'(T_n x_0, T_n x) + 1,$$

since $k \geq n \Rightarrow d'(T_n x_k, T_n x_{k+1}) < 2^{-k-1}$. So

$$n + 1 < d'(T_n x_0, T_n x_n) < d'(T_n x_0, T_n x) + 1 \Rightarrow d'(T_n x_0, T_n x) > n,$$

contradicting the pointwise-boundedness of \mathcal{F} . Q.E.D.

It is somewhat surprising that this result is best possible, in that it is impossible to weaken the condition that $\kappa(\alpha) > 0$ for $0 < \alpha < 1$, as the following example will indicate.

EXAMPLE 1. Let $X = [0, 1]$, $Y = R$. If $n \geq 2$, define

$$T_n(x) = 2n^2x, \quad 0 \leq x \leq 1/2n,$$

$$= 2n(1 - n)x + 2n - 1, \quad 1/2n \leq x \leq 1/n,$$

$$= (n/(1 - n))(x - 1), \quad 1/n \leq x \leq 1.$$

Each T_n is continuous, the family is pointwise-bounded (if $x > 0$, choose N with $x > 1/N$, then $n \geq N \Rightarrow T_n(x) \leq 1$), but the family is not uniformly bounded, as $T_n(1/2n) = n$.

We now compute $\psi(T_n, 1, 1/2n; \frac{1}{2})$. Since $(1+2n)/4n$ is the mid-point of $[1/2n, 1]$, we have $T_n((1+2n)/4n) = (2n-1)/4(n-1)$, and so $\psi(T_n, 1, 1/2n; \frac{1}{2}) = (2n-1)/4n(n-1)$; since $\kappa(\frac{1}{2}) \leq (2n-1)/4n(n-1)$ we see that $\kappa(\frac{1}{2}) = 0$. However, if $x \neq y$, we can show that $\psi(x, y; \alpha) > 0$ for $0 < \alpha < 1$. Assume $x < y$, if $1/N < x$ then, for $n \geq N$, T_n is linear on $[x, y]$, and so $n \geq N \Rightarrow \psi(T_n, x, y; \alpha) = \alpha$. Since

$$\psi(x, y; \alpha) = \min(\alpha, \psi(T_1, x, y; \alpha), \dots, \psi(T_n, x, y; \alpha))$$

(actually only those k ($1 \leq k \leq n$) for which $T_k x \neq T_k y$ are included in the minimum), $\psi(x, y; \alpha) > 0$. This example shows that it is impossible to weaken the condition $\kappa(\alpha) > 0$ in Theorem 1.

For a family of linear maps on a Banach space, uniform boundedness is equivalent to equicontinuity. This is not the case in general, as can be seen from the following example.

EXAMPLE 2. Define $\varphi: [0, 1] \rightarrow \mathcal{R}$ by $\varphi(x) = 2x$ if $0 \leq x \leq \frac{1}{2}$, $\varphi(x) = 2(1-x)$ if $\frac{1}{2} \leq x \leq 1$. Extend φ to \mathcal{R} by making $\varphi(x+1) = \varphi(x)$, $\forall x \in \mathcal{R}$. Define $T_n: [0, 1] \rightarrow \mathcal{R}$ by $T_n(x) = \varphi(2^n x)$; T_n is a sawtooth curve with 2^n teeth. Clearly $\{T_n | n=1, 2, \dots\}$ is a uniformly bounded family of continuous functions which is not equicontinuous. If $T_n x \neq T_n y$, let us compute $\psi(T_n, x, y; \alpha)$. If $(x, T_n x)$ and $(y, T_n y)$ lie on the same half of a tooth, then $\psi(T_n, x, y; \alpha) = \alpha$. If $(x, T_n x)$ and $(y, T_n y)$ lie on different teeth, or on different halves of the same tooth, there is a z lying between x and y such that $T_n z = T_n y$ and $(x, T_n x)$, $(z, T_n z)$ lie on the same half of a tooth. Since $\psi(T_n, x, z; \alpha) = \alpha$ and $T_n z = T_n y$, we see that $\psi(T_n, x, y; \alpha) > \alpha$. So for any $x \neq y$ in X , $\psi(x, y; \alpha) \geq \alpha$, and the hypotheses of Theorem 1 are fulfilled by a family of continuous functions which is not equicontinuous.

We conclude by making some remarks on the following problem: Suppose (X, d) is complete, (Y, d') is a metric space, and \mathcal{F} is a pointwise-bounded family of maps from X into Y such that each $T_\alpha \in \mathcal{F}$ is continuous on a closed subset S_α of X . Can we conclude that the family \mathcal{F} is uniformly bounded on a finite intersection $\bigcap_{k=1}^n S_{\alpha_k}$?

As in Theorem 1, the idea is to establish a contradiction by setting up some technical machinery. Assume that, if \mathcal{F} were not uniformly bounded on any finite intersection of the $\{S_\alpha\}$, we could establish the following assertion: Let $x_0 \in \bigcap_\alpha S_\alpha$, $p > 0$. Given $x_1, \dots, x_n \in X$, $T_1, \dots, T_n \in \mathcal{F}$ (with $T_k|_{S_k}$ continuous), $d(x_0, x_k) < p$, and given $M > 0$, then we can find $T_{n+1} \in \mathcal{F}$ and $x_{n+1} \in \bigcap_{k=1}^n S_k$ with $d(x_0, x_{n+1}) < p$, $d(x_n, x_{n+1}) < 1/2^{n+1}$, $d'(T_{n+1}x_{n+1}, T_{n+1}x_0) > M$ and $d'(T_k x_n, T_k x_{n+1}) < 1/2^{n+1}$ for $1 \leq k \leq n$. Such machinery leads to a contradiction in the same fashion as Theorem 1; the critical point here is that not only does $x_n \rightarrow x$, but for $k \geq n+1$ we have $x_k \in S_n$ and since $\{x_k | k \geq n+1\} \subseteq S_n$ and S_n is closed, we can use the continuity of T_n on S_n to complete the proof.

General conditions on \mathcal{F} and $\{S_\alpha\}$ are sufficiently involved as to limit their utility drastically. Although this theorem has been proved when X is a complete metric space with unique segments, the sets $\{S_\alpha\}$ are convex and \mathcal{F} consists of affine maps (this is done in [5]), these conditions are quite restrictive, and a theorem of this type for general X , $\{S_\alpha\}$, and \mathcal{F} appears to be extremely difficult to obtain.

Two questions naturally present themselves in connection with this paper. First, is it possible to establish reasonable sufficient conditions on \mathcal{F} and $\{S_\alpha\}$ as discussed previously? Second, is it possible to establish conditions for a pointwise-bounded family of continuous functions to be

equicontinuous? Answers to these questions would be extremely interesting.

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