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A VANISHING THEOREM FOR COHOMOLOGY

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ABSTRACT. A criterion is given for $H^0(G, A)=H^1(G, A)=0$, where G is a group and A is a G-module, in terms of the cohomology of a collection of subgroups of G.

In computing the cohomology of a group, knowledge of the cohomology of the subgroups is often useful. The following result is of that nature:

THEOREM. *Let $\mathscr{L}$ be a collection of subgroups of the group G and let A be a G-module. Assume that $\mathscr{L}$ has a minimum element and that G is generated by the subgroups in $\mathscr{L}$. It follows that if, for each L in $\mathscr{L}$,*

$$H^0(L, A) = H^1(L, A) = 0$$

then

$$H^0(G, A) = H^1(G, A) = 0.$$

PROOF. If L_0 is the minimum element of $\mathscr{L}$, then L_0 has no fixed points on A, as $H^0(L_0, A)=0$, so neither does G; hence $H^0(G, A)=0$. To show that $H^1(G, A)=0$ it suffices to prove that if

$$0 \to A \to E \to Z \to 0$$

is an exact sequence of G-modules, with Z the integers with trivial action, then the extension splits; this is because $H^1(G, A) \simeq \mathrm{Ext}^1_{ZG}(Z, A)$.

However, if $L \in \mathscr{L}$, then $H^1(L, A)=0$; so $E=A\oplus Z_L$, where Z_L is an L-module isomorphic to Z. Moreover, since $H^0(L, A)=0$, we have that $Z_L=E^L$, the fixed points of L in E. Since $L_0 \subseteq L$ we have $Z_L \subseteq Z_{L_0}$ so $Z_L=Z_{L_0}$ for every L in $\mathscr{L}$. But G is generated by all the subgroups L, so Z_{L_0} is also G-invariant and E splits over A as a G-module.

We conclude with an example which at the same time illustrates the use of this result and limits possible generalizations. Let $G=\mathrm{GL}(4, 2)$ and let

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A be a vector space of dimension four over GF(2) with G acting in the usual way. Let

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and set $L_0 = \langle S, T\rangle$, $L_1 = C_G(ST)$ and $L_2 = C_G(ST^{-1})$. It is easily verified, as $L_j \simeq \mathrm{GL}(2, 4)$, $j = 1, 2$, and GL(2, 4) has a center of odd order, that

$$H^i(L_k, A) = 0, \qquad i \geqq 0, k = 0, 1, 2.$$

Moreover, $L_1 \supseteq L_0$, $L_2 \supseteq L_0$ and $G = \langle L_0, L_1, L_2\rangle$. Hence, by our theorem $H^0(G, A) = H^1(G, A) = 0$, a very special case of a theorem of D. G. Higman [2]. However, $H^2(G, A) \neq 0$ as Blackburn [1] has proved.

References

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