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ON FUNCTIONS OF BOUNDED ROTATION

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ABSTRACT. For fixed $k \geqq 2$, denote by V_k and R_k the classes of functions regular in the unit disc and having boundary and radial rotation, respectively, at most $k\pi$. The concept of order of a function is defined for both V_k and R_k. For functions in these classes, the growth of integral and coefficient means is studied in terms of the order of the function. Some length-area results are also obtained.

1. **Introduction.** For fixed $k \geqq 2$, denote by V_k the class of normalized functions, analytic in the unit disc γ, which have boundary rotation at most $k\pi$. That is, a function f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

belongs to V_k if and only if $f'(z) \neq 0$ for $z \in \gamma$ and, with $z = re^{i\theta}$,

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{(zf'(z))'}{f'(z)} \right| d\theta \leqq k\pi. \tag{1.2}$$

Equivalently [7], $f \in V_k$ if and only if there exists a real-valued function μ on $[0, 2\pi]$ with $\int_0^{2\pi} d\mu(t) = 2$ and $\int_0^{2\pi} |d\mu(t)| \leqq k$ such that

$$f'(z) = \exp\left\{ -\int_0^{2\pi} \log(1 - ze^{-it})\, d\mu(t) \right\}. \tag{1.3}$$

Note that V_2 is the class of convex functions.

With $k \geqq 2$ still fixed, denote by R_k the class of functions g of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.4}$$

which have radial rotation at most $k\pi$. That is, $g \in R_k$ if and only if $g(z)/z \neq 0$

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for $z\in\gamma$ and, with $z=re^{i\theta}$,

$$\int_0^{2\pi}\left|\operatorname{Re}\frac{zg'(z)}{g(z)}\right|d\theta \leqq k\pi. \tag{1.5}$$

One can easily show that $g\in R_k$ if and only if there exists a real-valued function m on $[0, 2\pi]$ with $\int_0^{2\pi} dm(t)=2$ and $\int_0^{2\pi} |dm(t)|\leqq k$ such that

$$g(z) = z\exp\left\{-\int_0^{2\pi}\log(1-ze^{-it})\,dm(t)\right\}. \tag{1.6}$$

Note that R_2 is the class of starlike functions, and that $f\in V_k$ if and only if $g\in R_k$, where $g(z)=zf'(z)$.

We now proceed to define the concept of order for functions in V_k and R_k. Let $f\in V_k$ be given by (1.3). We shall require that μ be normalized by the conditions $\int_0^{2\pi}\mu(t)\,dt=0$ and $\mu(t)=(\mu(t+0)+\mu(t-0))/2$, in which case (1.3) defines a unique relationship between f and μ. Writing $\mu=\mu^+-\mu^-$ for the canonical decomposition of μ into the difference of nondecreasing functions, we define

$$\alpha(f) = \max\{\mu^+(t+0)-\mu^+(t-0) : t\in[0, 2\pi]\}$$

to be the *order* of f. Since μ is of bounded variation, $\alpha(f)$ exists and is the largest nonnegative jump of μ. Also note that if $\alpha(f)=\mu^+(\theta+0)-\mu^+(\theta-0)>0$, then the normalization condition on μ implies that μ^- is continuous at θ.

If $g\in R_k$ is given by (1.6), we shall require m to be normalized as above and write $m=m^+-m^-$ for the canonical decomposition of m. Now we define

$$\beta(g) = \max\{m^+(t+0)-m^+(t-0) : t\in[0, 2\pi]\}$$

to be the *order* of g. Again note that if $\beta(g)=m^+(\theta+0)-m^+(\theta-0)>0$, then m^- is continuous at θ. For $k=2$, one has Pommerenke's definition of the order of a starlike function [**9**].

In [**3**], F. Holland and D. K. Thomas obtained results concerning the order of a starlike function. Since every convex function is starlike, it is natural to ask whether analogues of the results in [**3**] are true for V_2, or more generally, for V_k and R_k. The purpose of this paper is to extend the results in [**3**] to V_k and R_k. It is interesting to note that to derive these results for V_k (and even for V_2), it seems necessary to consider the corresponding class R_k. The reason is that the representation (1.3) is for f', and many of the problems considered here seem to require a representation for f.

2. **Preliminary theorems.** If F is analytic in γ, set

$$M(r, F) = \max_{|z|=r} |F(z)|.$$

Also, with $z=re^{i\theta}$, set

$$L(r, F) = \int_0^{2\pi} |zF'(z)|\, d\theta$$

and

$$A(r, F) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{r} |F'(\rho e^{i\theta})|^2 \rho\, d\rho\, d\theta.$$

Then $L(r, F)$ is the length of the image of $\{z: |z|=r\}$ under F, and $\pi A(r, F)$ is the area of the image of $\{z: |z|\leqq r\}$ under F. Throughout this paper we shall use the notation f and $\alpha=\alpha(f)$ for a V_k function and its order, and g and $\beta=\beta(g)$ for an R_k function and its order.

THEOREM 2.1. (i) *If $g\in R_k$, then*

$$\lim_{r\to 1} \frac{\log M(r, g)}{-\log(1-r)} = \beta.$$

(ii) *If $f\in V_k$, then*

$$\lim_{r\to 1} \frac{\log M(r, f')}{-\log(1-r)} = \alpha,$$

$$\lim_{r\to 1} \frac{\log M(r, f)}{-\log(1-r)} = \max\{0, \alpha-1\}.$$

The proofs of (i) and the first part of (ii) are essentially the same as the proof of [**8**, Theorem 1]. The second statement in (ii) is proved in [**6**].

THEOREM 2.2. *If $g\in R_k$ and $\beta>0$, then*

$$\lim_{r\to 1} \frac{(1-r)M'(r, g)}{M(r, g)} = \beta,$$

where $M'(r, g)$ is the left derivative.

Since the proof is essentially the same as that of [**9**, Theorem 1], we again omit the details. There is, however, one point which should be mentioned. Suppose $\beta=m^+(\theta+0)-m^+(\theta-0)$. Since $\beta>0$, it follows that m^- is continuous at θ. This allows us to use the bounded convergence theorem in the same manner as Pommerenke, and the proof is then easily completed.

THEOREM 2.3. *If $g\in R_k$ is given by* (1.4), *then*

(i) $n\,|b_n|\, r^n \leqq 3kM(r, g)$,

(ii) $L(r, g)=O(1)M(r, g)$ *if* $\beta>0$.

The constant $O(1)$ depends on β, and hence on g.

PROOF. Since $g \in R_k$, $f \in V_k$ where $zf'(z) = g(z)$. Thus (i) follows immediately from [**10**, Theorem 1]. In the case $k=2$, Pommerenke [**9**] proved (ii) by using Theorem 2.2 above for $k=2$. His proof generalizes easily for $k>2$.

In our next theorem we derive a relationship between the classes V_k and R_k. This theorem will enable us to extend our results for R_k to the class V_k. We note first that any $f \in V_k$ is finitely valent [**2**], and in particular f has only finitely many zeros in γ.

THEOREM 2.4. *If $f \in V_k$, denote by $\{\delta_j\}_{j=0}^p$ the zeros of f in γ, where we choose $\delta_0 = 0$. Write $G(z) = cf(z)/h(z)$ where $c = \prod_{j=1}^p(-\delta_j)$ and h is given by $h(z) = \prod_{j=1}^p (z-\delta_j)$. Then there exists a constant K such that $G \in R_K$. Also, $\beta(G) = \max\{0, \alpha(f)-1\}$.*

REMARK. If f vanishes only at the origin (i.e. $p=0$), then by convention, $\prod_{j=1}^p = 1$.

PROOF. Since $f'(z) \neq 0$ for $z \in \gamma$, all zeros of f are simple, and so G vanishes only for $z=0$. Thus, from the definition of G, with $z = re^{i\theta}$,

$$\int_0^{2\pi} |d \arg G(z)|\, d\theta \leqq \int_0^{2\pi} |d \arg f(z)|\, d\theta + 2\pi p$$

for $r > \max_j \{|\delta_j|\}$. Since neither f nor f' vanish for $r > \max_j \{|\delta_j|\}$, a result of Biernacki [**1**] gives

$$\int_0^{2\pi} |d \arg f(z)|\, d\theta \leqq \int_0^{2\pi} \left| \operatorname{Re} \frac{(zf'(z))'}{f'(z)} \right| d\theta \leqq k\pi,$$

and so

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{zG'(z)}{G(z)} \right| d\theta = \int_0^{2\pi} |d \arg G(z)|\, d\theta \leqq (k+2p)\pi$$

for $r > \max_j \{|\delta_j|\}$. The first part of the theorem is now obvious. It is also clear from the definition of G that

$$\lim_{r\to 1} \frac{\log M(r, G)}{-\log(1-r)} = \lim_{r \to 1} \frac{\log M(r, f)}{-\log(1-r)},$$

and so from Theorem 2.1 we have $\beta(G) = \max\{0, \alpha(f)-1\}$.

3. **Integral and coefficient means.** In this section, if $\lambda > 0$ and F is analytic in γ we set, for $0 \leqq r < 1$,

$$I_\lambda(r, F) = \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^\lambda\, d\theta.$$

With this notation we have

THEOREM 3.1. *If $g \in R_k$ and $\lambda > 0$, then*

$$\lim_{r\to 1} \frac{\log I_\lambda(r, g)}{-\log(1-r)} = \max\{0, \beta\lambda - 1\}.$$

PROOF. From [4, p. 214] we have

$$I_\lambda(r, g) \geqq \frac{r^\lambda}{\pi} \int_0^r \frac{M(\rho, g)^\lambda}{\rho\lambda} \, d\rho. \tag{3.1}$$

It is clear from Theorem 2.1(i) that for any $\varepsilon > 0$, $M(r, g) \geqq B(\varepsilon)(1-r)^{-\beta+\varepsilon}$ and so from (3.1)

$$\liminf_{r\to 1} \frac{\log I_\lambda(r, g)}{-\log(1-r)} \geqq \max\{0, \beta\lambda - 1\}. \tag{3.2}$$

Next, as in [3], we note that with $z = re^{i\theta}$,

$$r(d/dr)\log|g(z)| = \operatorname{Re}(zg'(z)/g(z)),$$

and so

$$r\frac{d}{dr} I_\lambda(r, g) = \frac{\lambda}{2\pi} \int_0^{2\pi} |g(z)|^\lambda \operatorname{Re} \frac{zg'(z)}{g(z)} \, d\theta \leqq \frac{\lambda k}{2} M(r, g)^\lambda,$$

where we have used (1.5). Hence

$$I_\lambda(r, g) \leqq \frac{\lambda k}{2} \int_0^r \frac{M(\rho, g)^\lambda}{\rho} \, d\rho,$$

and so

$$\limsup_{r\to 1} \frac{\log I_\lambda(r, g)}{-\log(1-r)} \leqq \max\{0, \beta\lambda - 1\} \tag{3.3}$$

on using Theorem 2.1(i). Combining (3.2) and (3.3), we have Theorem 3.1.

THEOREM 3.2. *If $f \in V_k$ and $\lambda > 0$, then*

(i) $$\lim_{r\to 1} \frac{\log I_\lambda(r, f')}{-\log(1-r)} = \max\{0, \alpha\lambda - 1\}$$

and

(ii) $$\lim_{r\to 1} \frac{\log I_\lambda(r, f)}{-\log(1-r)} = \max\{0, \lambda\alpha - \lambda - 1\}.$$

PROOF. If $f \in V_k$ and g is defined by $g(z) = zf'(z)$, then $g \in R_k$ and $\beta(g) = \alpha(f)$, and so (i) follows immediately from Theorem 3.1. In order to prove (ii), let $G \in R_K$ be related to f as in Theorem 2.4, and denote the order of G

by $\beta=\max\{0, \alpha-1\}$. Then with the notation of Theorem 2.4, we have

$$I_\lambda(r,f) \leqq \prod_{j=1}^{p} (1 + r/|\delta_j|)^\lambda I_\lambda(r, G)$$

and, with $r>\max_j \{|\delta_j|\}$,

$$I_\lambda(r,f) \geqq \prod_{j=1}^{p} (r/|\delta_j| - 1)^\lambda I_\lambda(r, G),$$

from which the result follows upon using Theorem 3.1.

We now study the growth of the coefficients of V_k and R_k functions. If F given by $F(z)=\sum_{n=1}^{\infty} d_n z^n$ is analytic in γ, then as in [3] we define for $\lambda>0$ and $0\leqq r<1$,

$$P_\lambda(r, F) = \sum_{n=1}^{\infty} n^{\lambda-1} |d_n|^\lambda r^n.$$

Note that $P_2(r^2, F)=A(r, F)$. We now have

THEOREM 3.3. (i) *If* $g\in R_k$ *and* $\lambda>0$, *then*

$$\lim_{r\to 1} \frac{\log P_\lambda(r, g)}{-\log(1 - r)} = \beta\lambda.$$

(ii) If $f\in V_k$ *and* $\lambda>0$, *then*

$$\lim_{r\to 1} \frac{\log P_\lambda(r, f)}{-\log(1 - r)} = \lambda \max\{0, \alpha - 1\}.$$

PROOF. The proof of (i) is a direct analogue of that given in [3]. However, we include the proof, since we shall need the method to prove (ii). By Theorem 2.3(i),

$$P_\lambda(r^{\lambda+1}, g) = \sum_{n=1}^{\infty} (n\,|b_n|\, r^n)^\lambda \frac{r^n}{n}$$
$$\leqq (3kM(r, g))^\lambda \log \frac{1}{1 - r},$$

and so from Theorem 2.1(i),

$$\limsup_{r\to 1} \frac{\log P_\lambda(r, g)}{-\log(1 - r)} \leqq \beta\lambda. \tag{3.4}$$

Now suppose $\lambda\geqq 1$. By Hölder's inequality,

$$P_\lambda(r, g)\left(\log \frac{1}{1 - r}\right)^{\lambda-1} \geqq M(r, g)^\lambda,$$

and so Theorem 2.1(i) gives

$$\liminf_{r\to 1} \frac{\log P_\lambda(r, g)}{-\log(1 - r)} \geqq \beta\lambda,$$

which proves (i) for $\lambda\geqq 1$.

If $0<\lambda<1$, then again using Hölder's inequality,

$$\begin{aligned} P_1(r,g) &= \sum_{n=1}^{\infty}(n^{\lambda-1}|b_n|^{\lambda}r^n)^{\lambda}(n^{\lambda}|b_n|^{1+\lambda}r^n)^{1-\lambda} \\ &\leqq \left(\sum_{n=1}^{\infty} n^{\lambda-1}|b_n|^{\lambda}r^n\right)^{\lambda}\left(\sum_{n=1}^{\infty} n^{\lambda}|b_n|^{1+\lambda}r^n\right)^{1-\lambda} \\ &= P_{\lambda}(r,g)^{\lambda}(P_{1+\lambda}(r,g))^{1-\lambda}, \end{aligned}$$

and so

$$\frac{\log P_1(r,g)}{-\log(1-r)} \leqq \lambda\frac{\log P_{\lambda}(r,g)}{-\log(1-r)} + (1-\lambda)\frac{\log P_{1+\lambda}(r,g)}{-\log(1-r)},$$

which gives

$$\beta \leqq \lambda \liminf_{r\to 1}\frac{\log P_{\lambda}(r,g)}{-\log(1-r)} + (1-\lambda)(1+\lambda)\beta$$

on using (i) for the case $\lambda \geqq 1$. Hence

$$\beta\lambda \leqq \liminf_{r\to 1}\frac{\log P_{\lambda}(r,g)}{-\log(1-r)}, \tag{3.5}$$

and (i) follows for $0<\lambda<1$ upon combining (3.4) and (3.5).

In order to prove (ii), we first note that if $f\in V_k$ is given by (1.1), then **[10]**

$$n\,|a_n|\,r^n \leqq (1/2\pi)L(r,f) \leqq (k/2)M(r,f),$$

and so with the notation of Theorem 2.4,

$$n\,|a_n|\,r^n \leqq BM(r,G) \tag{3.6}$$

where $G\in R_K$ and $2B=k\max_{|z|\leqq 1}|h(z)/c|$. As in the proof of (i), we then find

$$P_{\lambda}(r^{\lambda+1},f) \leqq B^{\lambda}M(r,G)^{\lambda}\log\frac{1}{1-r},$$

which gives

$$\limsup_{r\to 1}\frac{\log P_{\lambda}(r,f)}{-\log(1-r)} \leqq \lambda\max\{0,\alpha-1\}.$$

In order to prove

$$\liminf_{r\to 1}\frac{\log P_{\lambda}(r,f)}{-\log(1-r)} \geqq \lambda\max\{0,\alpha-1\},$$

we proceed exactly as in the proof of (i), except that we use Theorem 2.1(ii) in place of Theorem 2.1(i). This proves Theorem 3.3.

To conclude this section, we have

THEOREM 3.4. (i) *If $g\in R_k$ is given by* (1.4), *then*

$$\limsup_{n\to\infty}\frac{\log^{+} n\,|b_n|}{\log n} = \beta.$$

(ii) *If* $f \in V_k$ *is given by* (1.1), *then*

$$\limsup_{n\to\infty} \frac{\log^+ n\,|a_n|}{\log n} = \max\{0, \alpha - 1\}.$$

The proof of this theorem is essentially the same as that given in [**3**, Theorem 5], so we omit the details. Note that (ii) follows immediately from (i) since $f \in V_k$ implies $g \in R_k$ where $g(z) = zf'(z)$.

4. **Some length-area results.** In this section we estimate $L(r, F)$ in terms of $A(r, F)^{1/2}$ for R_k and V_k. We must first prove a technical lemma.

LEMMA 1. (i) *If* $g \in R_k$, *then for* $0 \leqq r < 1$,

$$rM(r, g) \leqq 2^{k/2+1} M(r^2, g).$$

(ii) *If* $f \in V_k$ *and* $G \in R_K$ *is as in Theorem* 2,4, *then as* $r \to 1$, $A(r, G) = O(1)A(r, f)$.

PROOF. If $g \in R_k$, then $f \in V_k$ where $zf'(z) = g(z)$. It then follows immediately from [**5**, Corollary 3.2] that

$$\left(\frac{1-r}{1+r}\right)^{k/2+1} \frac{M(r, g)}{r}$$

is a decreasing function of r, which in turn gives (i).

With the notation of Theorem 2.4, $G(z) = cf(z)/h(z)$. If $p = 0$ in Theorem 2.4, then $c = 1$ and $h \equiv 1$. Write $z = re^{i\theta}$, and choose $r_0 > \max_j \{|\delta_j|\}$. Then with $r > r_0$, $|1/h(z)|$ and $|h'(z)/h^2(z)|$ are both bounded above, say by B. Hence

$$\left\{\int_0^{2\pi} |G'(z)|^2\, d\theta\right\}^{1/2} \leqq B\left\{\int_0^{2\pi} |f'(z)|^2\, d\theta\right\}^{1/2} + B\left\{\int_0^{2\pi} |f(z)|^2\, d\theta\right\}^{1/2}$$
$$\leqq 2B\left\{\int_0^{2\pi} |f'(z)|^2\, d\theta\right\}^{1/2},$$

and so

$$\text{(4.1)} \qquad \int_{r_0}^{r}\int_0^{2\pi} |G'(\rho e^{i\theta})|^2 \rho\, d\rho\, d\theta \leqq 4B^2 \int_{r_0}^{r}\int_0^{2\pi} |f'(\rho e^{i\theta})|^2 \rho\, d\rho\, d\theta.$$

Straightforward computation shows $\lim_{r\to 0} A(r, G)/A(r, f)$ exists and is finite, so there exists $B(r_0)$ such that $A(r, G) \leqq B(r_0) A(r, f)$ for $0 \leqq r \leqq r_0$. On combining this with (4.1), (ii) follows easily.

Using essentially the method of [**3**], we now prove

THEOREM 4.1. *If* $g \in R_k$, $\beta > 0$, *and* $\lambda \geqq 1$, *then as* $r \to 1$,

$$\int_0^{2\pi} |g'(re^{i\theta})|^\lambda\, d\theta = O(1) \frac{A(r, g)^{\lambda/2}}{(1-r)^{\lambda-1}}.$$

PROOF. Let $g \in R_k$ be given by (1.4), and write $A(r)$ for $A(r, G)$. Using Theorem 2.2, we see that there exists $B=B(\beta)$ and $r_0<1$ such that for $r>r_0$, $M(r^2, g) \leqq BA(r)^{1/2}$, and so from Lemma 1(i),

$$M(r, g) \leqq BA(r)^{1/2}. \tag{4.2}$$

However, from Theorem 2.3, $L(r, g)=O(1)M(r, g)$, and so $L(r, g)=O(1)A(r)^{1/2}$ as $r \to 1$, which gives Theorem 4.1 in the case $\lambda=1$.

We now suppose $\lambda>1$. From (1.5) it follows easily that with $z=re^{i\theta}$,

$$\frac{zg'(z)}{g(z)} = \frac{k+2}{4} p_1(z) - \frac{k-2}{4} p_2(z),$$

where $p_1, p_2 \in \mathscr{P}$, the class of normalized functions with positive real part. Using Minkowski's inequality and well-known properties of the class $\mathscr{P}$, we find that as $r \to 1$,

$$(1-r)^{\lambda-1} \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right|^\lambda d\theta = O(1). \tag{4.3}$$

Since

$$r^\lambda \int_0^{2\pi} |g'(z)|^\lambda \, d\theta \leqq M(r, g)^\lambda \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right|^\lambda d\theta,$$

(4.2) and (4.3) give, as $r \to 1$,

$$r^\lambda \int_0^{2\pi} |g'(z)|^\lambda \, d\theta = O(1)(1-r)^{1-\lambda} A(r)^{\lambda/2},$$

which proves the result.

In conclusion, we examine the same problem for V_k.

THEOREM 4.2. *If $f \in V_k$, $\alpha>1$, and $\lambda \geqq 1$, then as $r \to 1$,*

$$\int_0^{2\pi} |f'(re^{i\theta})|^\lambda \, d\theta = O(1) \frac{A(r, f)^{\lambda/2}}{(1-r)^{\lambda-1}}.$$

PROOF. With the notation of Theorem 2.4, $G(z) \cdot h(z)=cf(z)$, and so with $z=re^{i\theta}$,

$$\left\{ \int_0^{2\pi} |f'(z)|^\lambda \, d\theta \right\}^{1/\lambda} = O(1) \left\{ \int_0^{2\pi} |G'(z)|^\lambda \, d\theta \right\}^{1/\lambda} + O(1)M(r, G). \tag{4.4}$$

Since $G \in R_K$ and $\beta=\max\{0, \alpha-1\}>0$, we have from (4.2), (4.4), and

Theorem 4.1 that, as $r \to 1$,

$$\int_0^{2\pi} |f'(z)|^\lambda \, d\theta = O(1) \frac{A(r, G)^{\lambda/2}}{(1-r)^{\lambda-1}},$$

and upon using Lemma 1(ii), we find, as $r \to 1$,

$$\int_0^{2\pi} |f'(z)|^\lambda \, d\theta = O(1) \frac{A(r, f)^{\lambda/2}}{(1-r)^{\lambda-1}},$$

which proves the theorem.

COROLLARY 4.3. *If $f \in V_k$, then as $r \to 1$,*
(i) *$L(r,f) = O(1)A(r,f)^{1/2}$ if $\alpha > 1$,*
(ii) *$L(r,f) = O(1)A(r,f)^{1/2}(\log 1/(1-r))^{1/2}$ if $\alpha = 1$, and*
(iii) *$L(r,f)$ and $A(r,f)$ are both bounded if $\alpha < 1$.*

PROOF. (i) Let $\lambda = 1$ in Theorem 4.2.
(ii) From **[10]** we have $L(r,f) \leqq k\pi M(r,f)$. Also,

$$M(r^2, f) \leqq \sum_{n=1}^{\infty} |a_n| \, r^{2n} \leqq \left(\sum_{n=1}^{\infty} n \, |a_n|^2 \, r^{2n} \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{r^{2n}}{n} \right)^{1/2}$$
$$= A(r, f)^{1/2} \left(\log \frac{1}{1 - r^2} \right)^{1/2}.$$

From Theorem 2.4 and Lemma 1(i), it follows easily that $M(r,f) = O(1)M(r^2,f)$ as $r \to 1$, and hence (ii) is proved. The convex function f given by $f(z) = -\log(1-z)$, for which $\alpha = 1$, shows that the order of magnitude in (ii) is best possible for all $k \geqq 2$.

(iii) If $\alpha < 1$, it follows directly from (1.3) that f is bounded, and hence $L(r,f)$ and $A(r,f)$ are also bounded.

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