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SIMULTANEOUS APPROXIMATION AND INTERPOLATION IN l_1

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ABSTRACT. In a recent paper of R. Holmes and J. Lambert a geometrical approach was taken to the property of simultaneous approximation and interpolation which is norm preserving (SAIN), first introduced by F. Deutsch and P. Morris. An open question in both papers was if M is the subspace of l_1 consisting of the elements having only finitely many nonzero components does the triple (l_1, M, G) have property SAIN for all finite dimensional subspaces G contained in l_∞. This question is answered affirmatively by use of a generalization of Yamabe's theorem extending Helly's theorem.

0. **Introduction.** In [**1**] the concept of simultaneous approximation and interpolation which is norm preserving (SAIN) was introduced.

DEFINITION 0.1. Let X be a normed linear space, M a dense subset of X, L a finite dimensional subspace of X^*. The triple (X, M, L) has property SAIN if for every $x\in X$ and $\varepsilon>0$ there exists $y\in M$ such that $\|x-y\|<\varepsilon$, $\|x\|=\|y\|$ and $\lambda(x)=\lambda(y)$ for all $\lambda\in L$.

Many examples are given in [**1**] of various spaces X, subsets M, and subspaces L such that (X, M, L) is SAIN. In particular [**1**, Corollary 6.2], let M be the subspace of l_1 consisting of the elements having only finitely many nonzero components and let $L=\operatorname{span}\{y_i : i=1, 2, \cdots, n\}\subset l_\infty$. If each y_i is eventually constant, ($y_i=\{y_i(n)\}$ is eventually constant if there is an index N such that $y_i(N)=y_i(N+k)$ for all k) then (l_1, M, L) is SAIN.

In [**2**] a geometrical approach was taken to study the concept of SAIN and in particular it was found that [**2**, Corollary 3] if L is a finite dimensional subspace of c_0 (the predual of l_1) M as above, then (l_1, M, L) is SAIN.

It was conjectured that if L is any finite dimensional subspace of l_∞ then (l_1, M, L) is SAIN. In this paper that conjecture is verified.

In this paper the following notation and terminology is used. X denotes a real normed linear space; X^* the continuous dual of X, and $U(X)$ and $S(X)$ the closed unit ball and its boundary in X. A set $E\subset F$ is F-extremal, if whenever $tx+(1-t)y\in E$, $0<t<1$, and $x, y\in F$ then $x, y\in E$.

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An extremal subset consisting of a single point is called an extreme point. The extreme points of a set F will be denoted $\text{ext}(F)$. The convex hull of a set F is the intersection of all convex sets containing F and will be denoted $\text{co}(F)$. The closure of a set F will be denoted $\bar{F}$ or $\text{cl}(F)$.

The real valued function $\text{sgn}(\cdot)$: Reals$\to\{-1, 0, 1\}$ is defined via $\text{sgn}(x)=x/|x|$, $x\neq 0$ and $\text{sgn}(0)=0$.

1. *M*-core.

DEFINITION 1.1. Let F be a set in X, M a subset of F, then F is said to have M-core at f in F, if given m in M there exists $\delta(m,f)>0$ such that $(1+\delta)f-\delta m$ is in F, for $0\leqq\delta\leqq\delta(m,f)$. Let $e_i=\{\delta_{ij}\}$ denote the usual set of basis vectors in l_1.

PROPOSITION 1.1. *Let* $x\in S(l_1)$, $x=\sum_{i\in A} x_i e_i$; $x_i\neq 0$, $i\in A$, *A countable. Let F be minimal closed* $U(l_1)$ *extremal subset containing x. Let* $M=\text{co}(\text{sgn}(x_i)e_i{:}\,i\in A)$. *Then F has M-core at x.*

PROOF. By [**4**, Theorem 1] every closed bounded convex subset of l_1 is the closed convex hull of its extreme points. Since the extreme points of $U(l_1)$ are the usual basis vectors of norm one and F is $U(l_1)$ extremal it is easy to see that $F=\text{cl}(\text{co}(\text{sgn}(x_i)e_i{:}\,i\in A))$. Assume m in M is given. It remains to find $\delta(x, m)>0$ such that $z=(1+\delta)x-\delta m$ is in F. Since m is in M, $m=\sum_{i\in B} m_i\text{sgn}(x_i)e_i$, B a finite set. It suffices to find δ such that

$$\begin{aligned} 1 = \|z\| &= \sum_{i\in A}(1 + \delta)x_i - \delta m_i \\ &= (1 + \delta) \sum_{i\in A/B} |x_i| + \sum_{i\in B} |(1 + \delta)x_i - \delta m_i|. \end{aligned}$$

If one can insure that for $i\in B$, $\text{sgn}((1+\delta)x_i-\delta m_i)=\text{sgn}(x_i)$ then one would have

$$\begin{aligned} \sum_{i\in B}|(1 + \delta)x_i - \delta m_i| &= \sum_{i\in B}[(1 + \delta)x_i - \delta m_i]\text{sgn}(x_i) \\ &= \sum_{i\in B}(1 + \delta)x_i \,\text{sgn}(x_i) - \delta \sum_{i\in B} m_i \,\text{sgn}(x_i) \\ &= (1 + \delta)\sum_{i\in B} x_i - \delta \sum_{i\in B} m_i. \end{aligned}$$

Thus

$$\begin{aligned} \|z\| &= (1 + \delta) \sum_{i\in A/B} |x_i| + (1 + \delta)\sum_{i\in B}|x_i| - \delta\sum_{i\in B}|m_i| \\ &= (1 + \delta)\, \|x\| - \delta\, \|m\| = (1 + \delta) - \delta = 1. \end{aligned}$$

This would conclude the proof. To insure $\text{sgn}((1+\delta)x_i-\delta m_i)=\text{sgn}(x_i)$ for all $i\in B$, let $\varepsilon_i>0$ be chosen such that $|x_i|>\varepsilon_i|m_i|$. Let $\delta=\min_{i\in B}\{\varepsilon_i\}$.

Q.E.D.

2. Yamabe's theorem.

The following proposition is a version of Yamabe's theorem [**1**], [**4**] for convex sets.

PROPOSITION 2.1. *Let X be a Banach space, F a convex set contained in X. Let $y_1, \cdots, y_n \in X^*$. Let M be a norm dense convex subset of F. Let $f \in F$, F having M-core at f; then given $\varepsilon > 0$ there exists m in M such that $y_i(f) = y_i(m)$, $i = 1, \cdots, n$ and $\|f - m\| < \varepsilon$.*

PROOF. Let $K = F \cap \{b : \|b - f\| \leqq \varepsilon\}$. K is closed and convex, $N = M \cap K$ is convex and dense in K and f is an N core point for K. Define the continuous function $\varphi : F \to R^n$ via $\varphi(x) = (y_1(x), \cdots, y_n(x))$. Then φN is dense in φK and thus φN contains the relative interior of φK. Also φf is a φN core point of φK and this implies that φf is in the relative interior of φK, hence in φN. It follows that there is an $m \in N$ such that $\varphi m = \varphi f$. Q.E.D.

3. **Application to SAIN in l_1.**

DEFINITION 3.1. Let $x \in S(X)$; $F(x)$ denotes the minimal closed $U(X)$ extremal subset containing x.

THEOREM 3.1. *Let X be a Banach space, M a dense subspace in X. Then (X, M, L) is SAIN for all finite dimensional subspaces L in X^*, whenever $F(x) \cap M$ is dense in $F(x)$ and $F(x)$ has $F(x) \cap M$-core at x, for every $x \in S(X)$.*

PROOF. Given $x \in S(X)$, one applies Proposition 2.1 to $F(x)$ and obtains $m \in F(x) \cap M$ satisfying the SAIN conditions at x. The homogeneity of the functionals in L and of the norm yields the result for all $x \in X$. Q.E.D.

THEOREM 3.2. *Let $X = l_1$, M be the subspace of vectors with only finitely many nonzero coordinates. Then for all finite dimensional subspaces $L \subset l_\infty$ (X, M, L) is SAIN.*

PROOF. Given $x \in S(l_1)$; $F(x)$, $F(x) \cap M$ satisfy the hypotheses of Proposition 1.1. Hence, they satisfy the hypotheses of Theorem 3.1. Q.E.D.

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