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ON THE EMBEDDING $CP_n \subset CP_{2n}$

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In [1] we have proven that there are analytic embeddings of $CP_n \subset CP_m$ of arbitrary degree if $m > 2n$ while no such embeddings are possible for $m < 2n$ and only degree 2 (and, of course, 1) embeddings exist if $m = 2n$. Such embeddings were constructed for $n = 1, 2$, and 3. The explicit construction of degree 2 embeddings is sometimes useful since the restriction of the Hopf circle bundle over CP_{2n} to $CP_n \subset CP_{2n}$ is the real projective space $RP_{2n+1} \subset S^{4n+1}$. These embeddings give explicit (complex) normal bundles of RP_{2n+1} which are amenable to computations. The normal bundle of $CP_n \subset CP_{2n}$ is a stable complex vector bundle and is given by $(2n + 1)H^2 - (n + 1)H$ where H is the standard hopf bundle. It is not difficult to see that the normal bundles of $P^5 \subset S^{11}$ and $P^7 \subset S^{13}$ for these embeddings are $2\xi + 2\varepsilon$ and 6ε respectively (e.g. [2]).

The construction of degree 2 embeddings $CP_n \subset CP_{2n}$ came out of a conversation with P. Dupont to whom I am grateful for pointing out that CP_n can be viewed as the symmetric product of n copies of S^2. Thus a point in CP_n can be viewed as a nonzero polynomial

$$z_0 + z_1 t + \cdots + z_n t^n$$

with the usual homogeneity equivalence. This leads us to the following theorem:

THEOREM. *The map $q: CP_n \to CP_{2n}$ given by*

$$q(z_0 + z_1 t + \cdots + z_n t^n) = (z_0 + z_1 t + \cdots + z_n t^n)^2$$

is an analytic embedding of degree 2.

The proof is very easy—the map q is clearly of degree 2 and it is injective (1-1), since the square root of a polynomial (if it exists) is unique. It remains to show that q is of maximal rank. This is obvious once we note that the matrix of derivatives $(\partial q_i/\partial z_j)$ is

$$\begin{matrix} 2z_0 & 2z_1 & \cdots & & 2z_n & 0 & 0 & \cdots & 0 \\ 0 & 2z_0 & 2z_1 & \cdots & & 2z_n & 0 & \cdots & 0 \\ 0 & 0 & 2z_0 & 2z_1 & \cdots & & 2z_n & \cdots & 0 \\ 0 & \cdots & 0 & 2z_0 & 2z_1 & \cdots & & & 2z_n. \end{matrix}$$

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Taking the square matrix with the first nonzero z_i as diagonal entries we get a nonsingular one.

References

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2. E. Reese, *Embeddings of real projective spaces* (to appear).

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